Internat. J. Math. & Math. Sci.
Vol. 8 No. 4 (1985) 677-681

ON CERTAIN INEQUALITIES FOR SOME REGULAR FUNCTIONS IN $|z|<1$

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(Received August 6, 1985)

ABSTRACT. In this paper we give some inequalities for regular functions $f(z) = z+a_2z^2+\dots$ in $|z|<1$, especially for starlike and convex functions of order α, $0\leq\alpha<1$. To some extent those inequalities are the generalisations and improvements of the previous results given by Bernardi [1]. Some interesting consequences are given, too.

KEY WORDS AND PHRASES. *Regular functions, Starlike functions of order* α, *Convex functions of order* α, *Inequalities.*
1980 AMS SUBJECT CLASSIFICATION CODE. 30A32.

1. INTRODUCTION

Let A denote the class of functions $f(z) = z+a_2z^2+\dots$ which are regular in the unit disc $E=\{z:|z|<1\}$.

For a function $f\in A$ we say that it is starlike of order α, $0\leq\alpha<1$, in E, if and only if

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>\alpha\ ,\quad z\in E. \tag{1.1}$$

The class of such functions we denote by $S^*(\alpha)$. It is evident that $S^*(\alpha)\subseteq S^*$, where S^* is the class of starlike functions in E.

Similarly, for a function $f\in A$ we say that it is convex of order α, $0\leq\alpha<1$, in E, if and only if

$$\operatorname{Re}\{1+\frac{zf''(z)}{f'(z)}\}>\alpha\ ,\quad z\in E. \tag{1.2}$$

The class of such functions we denote by $K(\alpha)$. In this case we have also that $K(\alpha)\subseteq K(0)\equiv K$, where K is the class of convex functions in E. It is well-known that S^* and K, are the subclasses of the class S of univalent functions in E [2].

In this paper we give some inequalities of the integral type for the functions of A, especially for the functions of the classes $S^*(\alpha)$ and $K(\alpha)$, $0\leq\alpha<1$. For special cases of those inequalities we have some improved inequalities of Bernardi [1]. In this sense, our inequalities can be considered as the generalisations and improvements of the corresponding inequalities of Bernardi.

For the proofs of the coming inequalities we will use the following result of Miller [3]:

THEOREM A. Let $\Phi(u,v)$ be a complex function $\Phi: D\to C$ (C - complex plane, D - domain in $C\times C$), and let $u=u_1+iu_2$, $v=v_1+iv_2$. Suppose that the function Φ satisied the next conditions:

(a) $\Phi(u,v)$ is continuous in D;

(b) $(1,0)\in D$ and $\mathrm{Re}\Phi(1,0) > 0$;

(c) $\mathrm{Re}\Phi(u_2 i, v_1)\le 0$ for all $(u_2 i, v_1)\in D$ and such that $v_1 \le -(1/2)\cdot(1+u_2^2)$.

Let $p(z) = 1+p_1 z+\dots$ be regular in the unit disc E such that $(p(z), zp'(z))\in D$ for all $z\in E$. If

$$\mathrm{Re}\Phi(p(z), zp'(z)) > 0 , \quad z\in E,$$

then $\mathrm{Re}p(z)>0$, $z\in E$.

2. INEQUALITIES AND CONSEQUENCES

THEOREM 1. Let $f\in A$, $\alpha<1$ and $a>-1$. Then the following implication

$$\mathrm{Re}\,\frac{f(z)}{z}>\alpha \Rightarrow \mathrm{Re}\,\frac{a+1}{z^{a+1}}\int_0^z t^{a-1}f(t)dt>\alpha+\frac{1-\alpha}{3+2a}\ , \quad z\in E, \tag{2.1}$$

is true.

PROOF. Let's put

$$\alpha+\frac{1-\alpha}{3+2a} = \beta \quad \text{and} \quad \frac{a+1}{z^{a+1}}\int_0^z t^{a-1}f(t)dt = \beta + (1-\beta)p(z). \tag{2.2}$$

Then it is easily shown that $\beta<1$, p is regular in E and $p(0)=1$. From (2.2) we have that

$$\int_0^z t^{a-1}f(t)dt = \frac{z^{a+1}}{a+1}\,[\beta+(1-\beta)p(z)]\ ,$$

and hence, after differentiation and some simple transformations, we have that

$$\frac{f(z)}{z} - \alpha = \beta - \alpha + (1-\beta)p(z)+ \frac{1-\beta}{a+1}\, zp'(z). \tag{2.3}$$

Since $\mathrm{Re}\,\frac{f(z)}{z}>\alpha$, $z\in E$, then from (2.3) we get

$$\mathrm{Re}\,\{\beta-\alpha + (1-\beta)p(z) +\frac{1-\beta}{a+1}\, zp'(z)\}>0, \quad z\in E. \tag{2.4}$$

Now we may consider the function

$$\Phi(u,v) = \beta-\alpha + (1-\beta)u +\frac{1-\beta}{a+1}\, v$$

(it is noted $u=p(z)$, $v=zp'(z)$). It is directly checked that the function Φ satisfies the conditions (a), (b) and (c) of Theorem A. Namely, in this case it is $D=C^2$, Φ is continuous in D, $(1,0)\in D$, $\mathrm{Re}\Phi(1,0) = 1-\alpha>0$, while for all $(u_2 i, v_1)\in D$ such that $v_1\le-(1/2)\cdot(1+u_2^2)$ we have that

$$\mathrm{Re}\Phi(u_2 i, v_1) = \beta-\alpha + \frac{1-\beta}{a+1}\, v_1 \le \beta-\alpha+\frac{1-\beta}{a+1}\,[-\tfrac{1}{2}(1+u_2^2)] = -\frac{1}{2}\,\frac{1-\beta}{a+1}\, u_2^2\le 0.$$

Therefore, by applying Theorem A we have that $\mathrm{Re}p(z)>0$, $z\in E$; hence using (2.2) we finally get

$$\mathrm{Re}\,\frac{a+1}{z^{a+1}}\int_0^z t^{a-1}f(t)dt>\beta, \quad z\in E,$$

which was to be proved.

COROLLARY 1. If $f\in S^*(\alpha)$, $\frac{1}{2}\le\alpha<1$, then it is known [4] that

$$\mathrm{Re}\,\frac{f(z)}{z} > \frac{1}{3-2\alpha}\ , \quad z\in E,$$

and for those classes from Theorem 1 we have that

$$\mathrm{Re}\,\frac{a+1}{z^{a+1}}\int_0^z t^{a-1}f(t)dt>\frac{1}{3-2\alpha}+2\,\frac{1-\alpha}{(3-2\alpha)(3+2a)},\quad z\in E. \tag{2.5}$$

Because $f\in K(\alpha)\Leftrightarrow zf'(z)\in S^*(\alpha)$, $0\leqslant\alpha<1$, is true, then from (2.5) for $a=0$ we get the following estimates for $f\in K(\alpha)$, $\frac{1}{2}\leqslant\alpha<1$:

$$\mathrm{Re}\,\frac{f(z)}{z}>\frac{5-2\alpha}{3(3-2\alpha)} \tag{2.6}$$

COROLLARY 2. If $f\in K(\alpha)$, $0\leqslant\alpha<1$, then from [5] we have

$$\mathrm{Re}\,\frac{f(z)}{z}>\frac{1}{3-2\beta_1(\alpha)}$$

where

$$\beta_1(\alpha)=\begin{cases}\dfrac{2\alpha-1}{2(1-2^{1-2\alpha})}, & \alpha\neq\frac{1}{2}\\[2mm] \dfrac{1}{2\log 2}, & \alpha=\frac{1}{2}\end{cases} \tag{2.7}$$

and from Theorem 1

$$\mathrm{Re}\,\frac{a+1}{z^{a+1}}\int_0^z t^{a-1}f(t)dt>\frac{1}{3-2\beta_1(\alpha)}+2\,\frac{1-\beta_1(\alpha)}{(3-2\beta_1(\alpha))(3+2a)},\quad z\in E. \tag{2.8}$$

Especially, for $\alpha=0$, i.e. for $f\in K$, we have that $\beta_1(0)=\frac{1}{2}$ and from (2.8):

$$\mathrm{Re}\,\frac{a+1}{z^{a+1}}\int_0^z t^{a-1}\,f(t)dt>\frac{2+a}{3+2a},\quad z\in E, \tag{2.9}$$

which improves the earlier result of Bernardi [1], Th.8(A), where the constant or the right side of (2.9) is equal to 1/2 (namely, $\frac{2+a}{3+2a}>\frac{1}{2}$ for $a>-1$).

If $f\in A$ and $zf'(z)\in K(\alpha)$, $0\leqslant\alpha<1$, then from (2.8) for $a=0$ we have also that

$$\mathrm{Re}\,\frac{f(z)}{z}>\frac{5-2\beta_1(\alpha)}{3(3-2\beta_1(\alpha))},\quad z\in E, \tag{2.10}$$

where $\beta_1(\alpha)$ is defined by (2.7).

COROLLARY 3. If we put $zf'(z)$ in Theorem 1 instead of $f\in A$ we get that

$$\mathrm{Re}f'(z)>\alpha\Rightarrow\mathrm{Re}\,\frac{a+1}{z^{a+1}}\int_0^z t^a f'(t)dt>\alpha+\frac{1-\alpha}{3+2a},\quad z\in E, \tag{2.11}$$

is true.

From (2.11) for $a=0$ we have that

$$\mathrm{Re}f'(z)>\alpha\Rightarrow\mathrm{Re}\,\frac{f(z)}{z}>\frac{2\alpha+1}{3},\quad z\in E, \tag{2.12}$$

is true. From (2.12) by applying Theorem 1 once again we have that the following implication

$$\mathrm{Re}f'(z)>\alpha\Rightarrow\mathrm{Re}\,\frac{a+1}{z^{a+1}}\int_0^z t^{a-1}f(t)dt>\frac{2\alpha+1}{3}+\frac{2}{3}\cdot\frac{1-\alpha}{3+2a},\quad z\in E, \tag{2.13}$$

is true for $f\in A$, $\alpha<1$ and $a>-1$.

THEOREM 2. Let $f\in S^*(\alpha)$, $0\leqslant\alpha<1$, and let $a>\max\{-1,-2\alpha\}$, then we have that

$$\mathrm{Re}\,\frac{z^a f(z)}{\int_0^z t^{a-1}f(t)dt}>\frac{2a+2\alpha-1+\sqrt{(2a+2\alpha-1)^2+8(a+1)}}{4},\quad z\in E. \tag{2.14}$$

PROOF. The inequality (2.14) is equivalent to the inequality

$$\operatorname{Re} \frac{z^a f(z)}{(a+1)\int_0^z t^{a-1} f(t)dt} > \frac{2a+2\alpha-1+\sqrt{(2a+2\alpha-1)^2+8(a+1)}}{4(a+1)}, \quad z \in E. \qquad (2.14')$$

Let's put

$$\frac{z^a f(z)}{(a+1)\int_0^z t^{a-1} f(t)dt} = \beta + (1-\beta)p(z), \qquad (2.15)$$

where β is the number or the right side of the inequality (2.14'). It is easy to show that $0<\beta<1$, $p(z)$ is regular in E (moreover, it can be shown that the function $\frac{a+1}{z^a}\int_0^z t^{a-1} f(t)dt \in S^*(\alpha)$ if $f \in S^*(\alpha)$, $0 \leq \alpha < 1$, $a > -1$. Hence, the function $p(z)$ may have the removable singularity in $z=0$) and $p(0)=1$. From (2.15) after logarithmic differentiation we may get

$$\frac{zf'(z)}{f(z)} - \alpha = (a+1)\beta - a - \alpha + (a+1)(1-\beta)p(z) + (1-\beta)\frac{zp'(z)}{\beta+(1-\beta)p(z)},$$

and since $\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha$, $z \in E$, then

$$\operatorname{Re}\{(a+1)\beta - a - \alpha + (a+1)(1-\beta)p(z) + (1-\beta)\frac{zp'(z)}{\beta+(1-\beta)p(z)}\} > 0, \quad z \in E. \qquad (2.16)$$

In this case we consider the function

$$\Phi(u,v) = (a+1)\beta - a - \alpha + (a+1)(1-\beta)u + (1-\beta)\frac{v}{\beta+(1-\beta)u} \qquad (2.17)$$

The function $\Phi : D \to C$, where $D = (C \setminus \{-\frac{\beta}{1-\beta}\}) \times C$ and Φ is continuous in D. We have also that $(1,0) \in D$, $\operatorname{Re}\Phi(1,0) = 1-\alpha > 0$, while for all $(u_2 i, v_1) \in D$ so that $v_1 \leq -(1/2)(1+u_2^2)$ we get

$$\begin{aligned}\operatorname{Re}\Phi(u_2 i, v_1) &= (a+1)\beta - a - \alpha + (1-\beta)\beta \frac{v_1}{\beta^2+(1-\beta)^2 u_2^2} \\ &\leq (a+1)\beta - a - \alpha + (1-\beta)\beta \frac{-(1/2)(1+u_2^2)}{\beta^2+(1-\beta)^2 u_2^2} \\ &= -\frac{1}{\beta^2+(1-\beta)^2 u_2^2}[1+2(a+\alpha)-2(a+1)\beta]u_2^2\end{aligned}$$

(we used the fact that β is the solution of the equation $2(a+1)\beta^2-(2a+2\alpha-1)\beta-1=0$). To prove that $\operatorname{Re}\Phi(u_2 u, v_1) \leq 0$ we must show that $1+2(a+\alpha)-2(a+1)\beta \geq 0$, what is not difficult regarding the values for a,α and β. Therefore, from Theorem A and (2.16) we have that $\operatorname{Re}p(z) > 0$, $z \in E$, which is equivalent to (2.14'), i.e. (2.14).

COROLLARY 4. If $f \in K(\alpha)$, $0 \leq \alpha < 1$, what is equivalent to $zf'(z) \in S^*(\alpha)$, then from (2.14) for $a=0$ we have that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{2\alpha - 1 + \sqrt{(2\alpha-1)^2+8}}{4}, \quad z \in E. \qquad (2.18)$$

This is the former result given by Jack [6].

COROLLARY 5. If $f \in K$ then $\mathrm{Re}\, \frac{zf'(z)}{f(z)} > \frac{1}{2}$, [2], (which also implies from (2.18) for $\alpha = 0$) and because of that, from Theorem 2 we have

$$\mathrm{Re}\, \frac{z^a f(z)}{\int_0^z t^{a-1} f(t)dt} > \frac{a + \sqrt{a^2 + 2a + 2}}{2}, \quad z \in E, \tag{2.19}$$

which improves the former result of Bernardi [1], Th.7(B). Namely, the constant on the right side of (2.19) in [1] is $\frac{1+2a}{2}$, but

$$\frac{a+\sqrt{a^2+2a+2}}{2} > \frac{1+2a}{2} \quad (\Leftrightarrow \sqrt{(a+1)^2 + 1} > a + 1).$$

THEOREM 3. Let $f \in K(\alpha)$, $0 \leq \alpha < 1$, and $a > \max\{-1, -2\alpha\}$, then we have

$$\mathrm{Re}\, \frac{zf'(z)}{f(z) - \frac{a}{z^a} \int_0^z t^{a-1} f(t)dt} > \frac{2a+2\alpha-1+\sqrt{(2a+2\alpha-1)^2+8(a+1)}}{4}, \quad z \in E. \tag{2.20}$$

PROOF. If $f \in K(\alpha)$ then $zf'(z) \in S^*(\alpha)$, and by applying Theorem 2 to the function $zf'(z)$ and partial integration in corresponding integral we have the statement of Theorem 3.

COROLLARY 6. For $\alpha = 0$, i.e. for $f \in K$, we have that

$$\mathrm{Re}\, \frac{zf'(z)}{f(z) - \frac{a}{z^a} \int_0^z t^{a-1} f(t)dt} > \frac{2a - 1 + \sqrt{(2a-1)^2+8(a+1)}}{4}, \quad z \in E,$$

which also improves the result of Bernardi [1], Th.8(C).

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