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# HOLOMORPHIC EXTENSION OF GENERALIZATIONS OF $H^p$ FUNCTIONS

RICHARD D. CARMICHAEL

Department of Mathematical Sciences
New Mexico State University
Las Cruces, New Mexico 88003, U.S.A.

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ABSTRACT. In recent analysis we have defined and studied holomorphic functions in tubes in $\mathbb{C}^n$ which generalize the Hardy $H^p$ functions in tubes. In this paper we consider functions $f(z)$, $z = x+iy$, which are holomorphic in the tube $T^C = \mathbb{R}^n + iC$, where $C$ is the finite union of open convex cones $C_j$, $j = 1, \ldots, m$, and which satisfy the norm growth of our new functions. We prove a holomorphic extension theorem in which $f(z)$, $z \in T^C$, is shown to be extendable to a function which is holomorphic in $T^{O(C)} = \mathbb{R}^n + iO(C)$, where $O(C)$ is the convex hull of $C$, if the distributional boundary values in $\mathcal{S}'$ of $f(z)$ from each connected component $T^{C_j}$ of $T^C$ are equal.

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## 1. INTRODUCTION.

The purpose of this paper is to prove a holomorphic extension theorem (edge of the wedge theorem) for functions which are holomorphic in a tube in $\mathbb{C}^n$ and which satisfy a norm growth condition that generalizes the norm growth for $H^p$ functions in tubes. The basis for the analysis presented here is the analysis in our papers Carmichael [1-2].

We begin by stating some needed definitions. A set $C \subset \mathbb{R}^n$ is a cone (with vertex at the origin $\bar{0} = (0,0,\ldots,0)$ in $\mathbb{R}^n$) if $y \in C$ implies $\lambda y \in C$ for all positive scalars $\lambda$. A regular cone is an open convex cone $C$ such that $\bar{C}$ does not contain any entire straight line. The dual cone $C^*$ of a cone $C$ is defined as $C^* = \{t \in \mathbb{R}^n: \langle t,y\rangle \geq 0 \text{ for all } y \in C\}$; $C^*$ is always closed and convex (Vladimirov [3, p. 218]). The intersection of the cone $C$ with the unit sphere in $\mathbb{R}^n$ is called the projection of $C$ and is denoted $pr(C)$. The function

$$u_C(t) = \sup_{y \in pr(C)} (-\langle t,y\rangle)$$

is the indicatrix of the cone $C$, and we note that $C^* = \{t \in \mathbb{R}^n: u_C(t) \leq 0\}$. The set $T^C = \mathbb{R}^n + iC$ is a tube in $\mathbb{C}^n$. The convex hull (convex envelope) of a cone $C$ will be denoted by $O(C)$, and $O(C)$ is also a cone. Put $C_* = \mathbb{R}^n \setminus C^*$; the number

$$\rho_C = \sup_{t \in C_*} \frac{u_{O(C)}(t)}{u_C(t)}$$

characterizes the nonconvexity of the cone $C$ (Vladimirov [3, p. [illegible]]). Following Vladimirov [4, p. 930] we say that a cone $C \subset \mathbb{R}^n$ with interior points has an admissible set of vectors if there are vectors $e_k \in C$, $|e_k| = 1$, $k = 1,2, \ldots ,n$, which form a basis for $\mathbb{R}^n$; equivalently we say that such a set of $n$ vectors in $C$ is admissible for the cone $C$.

Let $B$ denote a proper open subset of $\mathbb{R}^n$. Let $0 < p < \infty$ and $A \geq 0$. Let $d(y)$ denote the distance from $y \in B$ to the complement of $B$ in $\mathbb{R}^n$. The space $S_A^p(T^B)$ (Carmichael [1, pp. 80-81]), $T^B = \mathbb{R}^n + iB$, is the set of all functions $f(z)$, $z = x + iy \in T^B$, which are holomorphic in $T^B$ and which satisfy

$$||f(x+iy)||_{L^p} = \left(\int_{\mathbb{R}^n} |f(x+iy)|^p \, dx\right)^{1/p} \leq \tag{1.1}$$
$$\leq M\,(1 + (d(y))^{-r})^s \exp(2\pi A|y|), \quad y \in B,$$

for some constants $r \geq 0$ and $s \geq 0$ which can depend on $f$, $p$, and $A$ but not on $y \in B$ and for some constant $M = M(f,p,A,r,s)$ which can depend on $f$, $p$, $A$, $r$, and $s$ but not on $y \in B$. We defined and studied the functions $S_A^p(T^B)$ in Carmichael [1-2]. The spaces $S_A^p(T^B)$ were defined to generalize the $H^p$ functions in tubes (Stein and Weiss [5, Chapter III]) and to contain the previous generalizations of the $H^p$ functions of Vladimirov [6] and Carmichael and Hayashi [7].

We proved in Carmichael [1, Theorem 4.1, p. 92] that if $B$ is a proper open connected subset of $\mathbb{R}^n$ then any element $f(z) \in S_A^p(T^B)$, $1 < p \leq 2$, $A \geq 0$, has a Fourier-Laplace integral representation for $z \in T^B$ in terms of a function $g(t)$ which satisfies certain norm growth properties. In addition we proved in Carmichael [1, Corollary 4.1, p. 93] that if $B = C$, an open convex cone in $\mathbb{R}^n$, then $f(x+iy)$ has a unique boundary value as $y \to \bar{0}$, $y \in C$, in the strong topology of $\mathcal{S}'$, the space of tempered distributions.

In this paper we prove a holomorphic extension theorem (edge of the wedge theorem) for holomorphic functions in $T^C$ which satisfy (1.1) for $y \in C$ where $C$ is a finite union of open convex cones in $\mathbb{R}^n$; the extended function is holomorphic in $T^{O(C)}$ where $O(C)$ is the convex hull of $C$. To obtain our extension theorem we use the information from Carmichael [1] which is contained in the preceding paragraph.

We proceed to the result of this paper after making the following definition; the subspace $\mathcal{S}'_p$ of $\mathcal{S}'$, $1 \leq p < \infty$, is defined to be the set of all measurable functions $g(t)$, $t \in \mathbb{R}^n$, such that there exists a real number $b \geq 0$ for which $((1 + |t|^p)^{-b}\, g(t)) \in L^p$ (Carmichael [1, p. 83]).

All subsequent notation and terminology in this paper are that of Carmichael [1-2].

## 2. HOLOMORPHIC EXTENSION.

Let $C$ be an open cone in $\mathbb{R}^n$ such that $C = \bigcup_{j=1}^{m} C_j$ where the $C_j$, $j = 1, \ldots ,m$, are open convex cones in $\mathbb{R}^n$ and $m$ is a positive integer. Let $f(z)$ be holomorphic in the tubular cone $T^C = \mathbb{R}^n + iC$ and satisfy (1.1) for $y \in C$ and for $1 < p \leq 2$. For any $y \in C_j$, $j = 1, \ldots ,m$, the distance from $y$ to the boundary of $C$ is larger than or equal to the distance from $y$ to the boundary of $C_j$ from which it follows that $f(z) \in S_A^p(T^{C_j})$, $1 < p \leq 2$, $j = 1, \ldots ,m$. Thus by Carmichael [1, Corollary 4.1, p. 93] there exist measurable functions $g_j(t) \in \mathcal{S}'_q$, $(1/p) + (1/q) = 1$, with $\mathrm{supp}(g_j) \subseteq \{t: u_{C_j}(t) \leq A\}$

almost everywhere such that

$$f(z) = \int_{R^n} g_j(t) \exp(2\pi i\langle z,t\rangle)\, dt, \quad z \in T^{C_j}, \; j = 1, \ldots, m, \tag{2.1}$$

pointwise and

$$\lim_{\substack{y\to\bar{0} \\ y\in C_j}} f(x+iy) = \mathfrak{F}[g_j] \in \mathcal{S}', \; j = 1, \ldots, m, \tag{2.2}$$

in the strong topology of $\mathcal{S}'$ with $\mathfrak{F}[g_j]$ being the $\mathcal{S}'$ Fourier transform of $g_j \in \mathcal{S}'_q \subset \mathcal{S}'$.

We now state and prove the main result of this paper.

THEOREM. Let C be an open cone in $\mathbb{R}^n$ which is the union of a finite number of open convex cones, $C = \bigcup_{j=1}^{m} C_j$, such that $(0(C))^*$ contains interior points and has an admissible set of vectors. Let $f(z)$, $z = x+iy$, be holomorphic in the tubular cone $T^C$ and satisfy (1.1) for $y \in C$ and $1 < p \leq 2$. Let the boundary values of $f(x+iy)$ in the strong topology of $\mathcal{S}'$ corresponding to each connected component $C_j$, $j = 1, \ldots, m$, of C given in (2.2) be equal in $\mathcal{S}'$. Then there is a function $F(z)$ which is holomorphic in $T^{0(C)}$ and which satisfies $F(z) = f(z)$, $z \in T^C$, where $F(z)$ is of the form

$$F(z) = P(z)\, H(z), \quad z \in T^{0(C)},$$

with $P(z)$ being a polynomial in z and $H(z) \in S^2_{A\rho_C}(T^{0(C)}) \cap S^q_{A\rho_C}(T^{0(C)})$, $(1/p) + (1/q) = 1$.

PROOF. By hypothesis the boundary values in (2.2) above are equal in $\mathcal{S}'$. Since the Fourier transform is a topological isomorphism of $\mathcal{S}'$ onto $\mathcal{S}'$ we have that the elements $g_j(t) \in \mathcal{S}'_q \subset \mathcal{S}'$, $(1/p) + (1/q) = 1$, $j = 1, \ldots, m$, obtained in the first paragraph of this section satisfy

$$g_1(t) = g_2(t) = \ldots = g_m(t) \tag{2.3}$$

in $\mathcal{S}'$. We call this common value $g(t)$ and have $g(t) \in \mathcal{S}'_q$, $(1/p) + (1/q) = 1$. Now

$$u_C(t) = \max_{j=1,\ldots,m} u_{C_j}(t), \quad t \in \mathbb{R}^n. \tag{2.4}$$

We have $u_C(t) = u_{0(C)}(t)$, $t \in C^*$, (Vladimirov [3, p. 219, (54)]); and from the definition of $\rho_C$ we have $u_{0(C)}(t) \leq \rho_C\, u_C(t)$, $t \in C_* = \mathbb{R}^n \setminus C^*$. Since $1 \leq \rho_C < \infty$ (Vladimirov [3, p. 220]) here we have $u_{0(C)}(t) \leq \rho_C\, u_C(t)$, $t \in \mathbb{R}^n$. From (2.4) we now obtain

$$u_{0(C)}(t) \leq \rho_C \max_{j=1,\ldots,m} u_{C_j}(t), \quad t \in \mathbb{R}^n. \tag{2.5}$$

From (2.3) and the fact that $\text{supp}(g_j) \subseteq \{t: u_{C_j}(t) \leq A\}$ almost everywhere, $j = 1, \ldots, m$, we have that $g \in \mathcal{S}'_q \subset \mathcal{S}'$ vanishes on $\bigcup_{j=1}^{m} \{t: u_{C_j}(t) > A\}$ as a distribution. Now let $t \in \{t: u_{0(C)}(t) > A\rho_C\}$; for such a point t we have by (2.5) that

$$A\rho_C < u_{0(C)}(t) \leq \rho_C \max_{j=1,\ldots,m} u_{C_j}(t)$$

and hence

$$\max_{j=1,\ldots,m} u_{C_j}(t) > A.$$

Thus if $t \in \{t: u_{O(C)}(t) > A\rho_C\}$ then $t \in \bigcup_{j=1}^{m} \{t: u_{C_j}(t) > A\}$ and on this latter set $g$ vanishes. Since $\{t: u_{O(C)}(t) \leq A\rho_C\}$ is a closed set in $\mathbb{R}^n$ we thus have

$$\operatorname{supp}(g) \subseteq \{t: u_{O(C)}(t) \leq A\rho_C\} \tag{2.6}$$

in $\mathcal{S}'$ and $\{t: u_{O(C)}(t) \leq A\rho_C\} = (O(C))^* + \overline{N(\bar{0};A\rho_C)}$ (Vladimirov [4, Lemma 1, p. 936]) with $\overline{N(\bar{0};A\rho_C)}$ being the closure of the open ball in $\mathbb{R}^n$ centered at $\bar{0}$ and with radius $A\rho_C$. Recall from section 1 that the dual cone $(O(C))^*$ is closed and convex and by hypothesis in this Theorem $(O(C))^*$ contains interior points and has an admissible set of vectors. Since $g \in \mathcal{S}_q' \subset \mathcal{S}'$ has order 0 then by Vladimirov [4, Theorem 1, p. 930]

$$g(t) = \prod_{k=1}^{n} \langle e_k, \text{gradient}\rangle^2 \; G(t) \tag{2.7}$$

where $\{e_k\}_{k=1}^n$ is an admissible set of vectors for the cone $(O(C))^*$, $G(t)$ is a continuous function of $t \in \mathbb{R}^n$ which is unique corresponding to $\{e_k\}_{k=1}^n$ and the order 0 of $g \in \mathcal{S}_q' \subset \mathcal{S}'$, $\operatorname{supp}(G) \subseteq \{t: u_{O(C)}(t) \leq A\rho_C\} = (O(C))^* + \overline{N(\bar{0};A\rho_C)}$, and

$$|G(t)| \leq K\,(1 + |t|), \quad t \in \mathbb{R}^n, \tag{2.8}$$

where the constant $K$ is independent of $t \in \mathbb{R}^n$. (In Vladimirov [4, Theorem 1, p. 930] the term "acute" in our present situation means that $((O(C))^*)^* = \overline{O(C)}$ (Vladimirov [3, p. 218]) should have non-empty interior (Vladimirov [4, p. 930]) which is certainly the case in this Theorem.) Since $G(t)$ is continuous on $\mathbb{R}^n$, then $\operatorname{supp}(G) \subseteq \{t: u_{O(C)}(t) \leq A\rho_C\}$ as a function (Schwartz [8, Chapter 1, sections 1 and 3]). (This fact is also obtained in the proof of Vladimirov [4, Theorem 1], and the containment $\operatorname{supp}(G) \subseteq \{t: u_{O(C)}(t) \leq A\rho_C\}$ which is stated preceding to (2.8) gives the support of $G(t)$ as a function.) We now choose a function $\lambda(t) \in C^\infty$, $t \in \mathbb{R}^n$, such that for any n-tuple $\alpha$ of nonnegative integers $|D^\alpha\lambda(t)| \leq M_\alpha$, $t \in \mathbb{R}^n$, where $M_\alpha$ is a constant which depends only on $\alpha$; and for $\epsilon > 0$, $\lambda(t) = 1$ for $t$ on an $\epsilon$ neighborhood of $\{t: u_{O(C)}(t) \leq A\rho_C\}$ and $\lambda(t) = 0$ for $t \in \mathbb{R}^n$ but not on a $2\epsilon$ neighborhood of $\{t: u_{O(C)}(t) \leq A\rho_C\}$ (Carmichael [1, p. 94]). We have that $(\lambda(t)\exp(2\pi i\langle z,t\rangle)) \in \mathcal{S}$ as a function of $t \in \mathbb{R}^n$ for $z \in T^{O(C)}$. Recalling (2.6) we now put

$$F(z) = \int_{\mathbb{R}^n} g(t)\exp(2\pi i\langle z,t\rangle)\,dt = \int_{\mathbb{R}^n} g(t)\,\lambda(t)\exp(2\pi i\langle z,t\rangle)\,dt, \quad z \in T^{O(C)}. \tag{2.9}$$

From (2.7) and $\operatorname{supp}(G) \subseteq \{t: u_{O(C)}(t) \leq A\rho_C\}$ as a function we have (Vladimirov [4, (3.1), p. 931])

$$F(z) = \left(\prod_{k=1}^{n} \langle e_k, -2\pi i z\rangle^2\right) H(z), \; z \in T^{O(C)}, \tag{2.10}$$

where

$$H(z) = \int_{\{t: u_{O(C)}(t) \leq A\rho_C\}} G(t)\exp(2\pi i\langle z,t\rangle)\,dt, \; z \in T^{O(C)}. \tag{2.11}$$

From the continuity of $G(t)$ and (2.8) we easily have $G(t) \in \mathcal{S}_p'$ for all $p$, $1 \leq p < \infty$; this combined with the support of $G(t)$ as a function and Carmichael [1, Theorem 6.1, p. 98] yield

$$(\exp(-2\pi\langle y,t\rangle)\,G(t)) \in L^p, \; y \in O(C), \tag{2.12}$$

and

$$||\exp(-2\pi<y,t>)\ G(t)||_{L^p} \leq M\ (1 + (d(y))^{-r})^s\ \exp(2\pi A\,\rho_C\,|y|),\quad y \in O(C)\ , \tag{2.13}$$

for constants $r = r(G,p,A) \geq 0$, $s = s(G,p,A) \geq 0$, and $M = M(G,p,A,r,s) > 0$, which are independent of $y \in O(C)$, and for all p, $1 \leq p < \infty$. Then (2.12), (2.13), and Carmichael [1, Theorem 5.1, p. 97] prove $H(z) \in S^q_{A\,\rho_C}(T^{O(C)})$, $(1/p) + (1/q) = 1$, for all p, $1 < p \leq 2$, and in particular $H(z) \in S^2_{A\,\rho_C}(T^{O(C)})$. Then by (2.10), F(z) defined in (2.9) is holomorphic in $T^{O(C)}$, and of course (2.10) is the desired representation of F(z) in the statement of the Theorem where the polynomial P(z) is

$$P(z) = \prod_{k=1}^{n} <e_k,\ -2\pi i z>^2$$

and $H(z) \in S^2_{A\,\rho_C}(T^{O(C)}) \cap S^q_{A\,\rho_C}(T^{O(C)})$, $(1/p) + (1/q) = 1$, is given in (2.11). By (2.3), (2.6), and the definition of $\lambda(t)$ preceding (2.9), we see that (2.1) can be rewritten as

$$f(z) = \int_{\mathbb{R}^n} g(t)\ \lambda(t)\ \exp(2\pi i<z,t>)\ dt =$$

$$= \int_{\mathbb{R}^n} g(t)\ \exp(2\pi i<z,t>)\ dt,\quad z \in T^{C_j},\quad j = 1,\dots,m\,.$$

These identities and (2.9) show that F(z) is the desired holomorphic extension of f(z) to $T^{O(C)}$ and $F(z) = f(z)$, $z \in T^C$. The proof of the Theorem is complete.

We emphasize that cones C exist for which the hypotheses of the Theorem are satisfied corresponding to C and $(O(C))^*$, and examples are easily constructed. If O(C) in the Theorem is regular (i.e. if $\overline{O(C)}$ does not contain an entire straight line in this case since O(C) is open and convex) then the interior of $(O(C))^*$ is not empty; the Theorem applies in this case if $(O(C))^*$ has an admissible set of vectors.

In the Theorem we have desired to obtain a result in which the holomorphic extension function could be represented in terms of an $S^p_{A\,\rho_C}(T^{O(C)})$ space; this happens under the assumptions on $(O(C))^*$ in the Theorem. Under these assumptions we were able to conclude that the continuous function G(t) in the representation (2.7) had pointwise support in $\{t\colon u_{O(C)}(t) \leq A\,\rho_C\}$. From this fact we were able to use Carmichael [1, Theorem 6.1] and then Carmichael [1, Theorem 5.1] to obtain that H(z) in (2.11) belongs to $S^q_{A\,\rho_C}(T^{O(C)})$, $(1/p) + (1/q) = 1$, for all p, $1 < p \leq 2$; and hence the desired representation of the holomorphic extension function F(z) was obtained in (2.10).

From the proof of the Theorem the common value $g(t) \in \mathcal{L}'_q$, $(1/p) + (1/q) = 1$, $1 < p \leq 2$, in (2.3) has $\mathrm{supp}(g) \subseteq \{t\colon u_{O(C)}(t) \leq A\,\rho_C\}$ in $\mathcal{L}'$ (recall (2.6)). If supp(g) is contained in this set almost everywhere as a function as well then the restrictions on $(O(C))^*$ in the Theorem can be deleted in obtaining a holomorphic extension result as we show in the following corollary.

COROLLARY 1. Let C be an open cone in $\mathbb{R}^n$ which is the union of a finite number of open convex cones, $C = \bigcup_{j=1}^{m} C_j$. Let f(z), $z = x+iy$, be holomorphic in the tubular cone $T^C$ and satisfy (1.1) for $y \in C$ and $1 < p \leq 2$. Let the boundary values of $f(x+iy)$ in the strong topology of $\mathcal{L}'$ corresponding to each connected component $C_j$,

$j = 1, \ldots, m$, of $C$ given in (2.2) be equal in $\mathcal{D}'$ and let this common value $g(t)$ have support in $\{t: u_{O(C)}(t) \leq A\rho_C\}$ almost everywhere (as well as in $\mathcal{D}'$). Then there is a function $F(z)$ which is holomorphic in $T^{O(C)}$ and which satisfies $F(z) = f(z)$, $z \in T^C$; and if $p = 2$, $F(z) \in S^2_{A\rho_C}(T^{O(C)})$.

PROOF. Proceeding as in the proof of the Theorem we obtain the common value $g(t) \in \mathcal{D}'_q$, $(1/p) + (1/q) = 1$, from (2.3) and $\text{supp}(g) \subseteq \{t: u_{O(C)}(t) \leq A\rho_C\}$ in $\mathcal{D}'$. By our assumption $\text{supp}(g) \subseteq \{t: u_{O(C)}(t) \leq A\rho_C\}$ almost everywhere; thus by Carmichael [1, Theorem 6.1, p. 98], $g(t)$ satisfies

$$(\exp(-2\pi\langle y,t\rangle)\, g(t)) \in L^q, \quad y \in O(C), \tag{2.14}$$

and

$$\|\exp(-2\pi\langle y,t\rangle)\, g(t)\|_{L^q} \leq M\,(1 + (d(y))^{-r})^s \exp(2\pi A\rho_C|y|), \quad y \in O(C), \tag{2.15}$$

for constants $r = r(g,q,A) \geq 0$, $s = s(g,q,A) \geq 0$, and $M = M(g,q,A,r,s) > 0$ which are independent of $y \in O(C)$. Then by Carmichael [1, Theorem 3.1, pp. 84-85] the function

$$F(z) = \int_{\mathbb{R}^n} g(t)\exp(2\pi i\langle z,t\rangle)\,dt = \int_{\mathbb{R}^n} g(t)\,\lambda(t)\exp(2\pi i\langle z,t\rangle)\,dt, \quad z \in T^{O(C)}, \tag{2.16}$$

is holomorphic in $T^{O(C)}$ where $\lambda(t) \in C^\infty$ is the function defined in the proof of the Theorem. As in the proof of the Theorem $F(z)$ is the desired holomorphic extension of $f(z)$ to $T^{O(C)}$. If $p = 2$ then $q = 2$; in this case (2.14), (2.15), and Carmichael [1, Theorem 5.1, p. 97] yield that $F(z) \in S^2_{A\rho_C}(T^{O(C)})$. The proof is complete.

We have a more general holomorphic extension theorem than either the Theorem or Corollary 1. Here $O(C)$ is as general as possible and we make no assumption on the constructed $g(t)$ in (2.3). We lose the explicit information on $F(z)$ being in an $S^p_{A\rho_C}(T^{O(C)})$ space however.

COROLLARY 2. Let the open cone $C$ and the function $f(z)$ be as in the hypothesis of Corollary 1 with $1 < p \leq 2$. Let the boundary values of $f(x+iy)$ in the strong topology of $\mathcal{D}'$ corresponding to each connected component $C_j$, $j = 1, \ldots, m$, of $C$ given in (2.2) be equal in $\mathcal{D}'$. Then there is a holomorphic function $F(z)$ in $T^{O(C)}$ such that $F(z) = f(z)$, $z \in T^C$.

PROOF. Define $F(z)$, $z \in T^{O(C)}$, as in (2.16) where $g \in \mathcal{D}'_q \subset \mathcal{D}'$, $(1/p) + (1/q) = 1$, is the common value in (2.3) in $\mathcal{D}'$ and $\text{supp}(g) \subseteq \{t: u_{O(C)}(t) \leq A\rho_C\}$ in $\mathcal{D}'$ from the proof of the Theorem. Then $F(z)$ is holomorphic in $T^{O(C)}$ by the necessity of Vladimirov [3, Theorem 2, p. 239] and is the desired holomorphic extension of $f(z)$ to $T^{O(C)}$ because of (2.3) and (2.1). (Recall the proof of the Theorem.) The proof is complete.

Notice from Vladimirov [3, Theorem 2, p. 239] that $F(z)$ in Corollary 2 does satisfy a pointwise growth estimate; but we cannot conclude that $F(z)$ is in an $S^p_{A\rho_C}(T^{O(C)})$ space for any $p$ in Corollary 2.

In the Theorem and Corollaries 1 and 2 the holomorphic extension function $F(z)$, $z \in T^{O(C)}$, is defined by (2.9) (i.e. (2.16)) where $g(t) \in \mathcal{D}'_q \subset \mathcal{D}'$, $(1/p) + (1/q) = 1$, and $\text{supp}(g) \subseteq \{t: u_{O(C)}(t) \leq A\rho_C\}$ in $\mathcal{D}'$. Since $O(C)$ is an open convex cone then in each of the results we can also conclude that

$$\lim_{\substack{y \to \bar{0} \\ y \in O(C)}} F(x+iy) = \mathfrak{F}[g] \in \mathcal{S}' \tag{2.17}$$

in the strong topology of $\mathcal{S}'$ by the boundary value proof in Carmichael [1, Corollary 4.1, p. 93]; here $\mathfrak{F}[g]$ is the $\mathcal{S}'$ Fourier transform. Further, if $O(C)$ is a regular cone, $A = 0$, and $p = 2$, in Corollary 1 then we can conclude in Corollary 1 that

$$F(z) = \langle \mathfrak{F}[g], K(z-t) \rangle = \langle \mathfrak{F}[g], Q(z\,;\,t) \rangle, \quad z \in T^{O(C)}, \tag{2.18}$$

in $\mathcal{S}'$ by Carmichael [1, Corollary 4.2, p. 94] where $\mathfrak{F}[g]$ is the boundary value in (2.17) and $K(z-t)$ and $Q(z\,;\,t)$ are the Cauchy and Poisson kernels (Carmichael [1, p. 83]), respectively, corresponding to the tube $T^{O(C)}$. (Recall from the sentence preceding the statement of Carmichael [1, Corollary 4.2, p. 94] that $g \in \mathcal{S}'_2$ implies $\mathfrak{F}[g] \in \mathcal{D}'_{L^2} \subset \mathcal{S}'$.)

If the cone $C$ is $(0,\infty)$ or $(-\infty,0)$ or $(-\infty,0) \cup (0,\infty)$ in 1 dimension then of course $d(y) = |y|$, $y \in C$, in (1.1). We have the following interesting result in 1 dimension for $C = (-\infty,0) \cup (0,\infty)$. Note that $(O(C))^* = \{0\}$ here which does not have interior points; so the following result is like Corollary 2.

COROLLARY 3. Let $f(z)$ be holomorphic in $\mathbb{R}^1 + iC$, $C = (-\infty,0) \cup (0,\infty)$, and satisfy (1.1) for $1 < p \leq 2$. Let the boundary values of $f(x+iy)$ in the strong topology of $\mathcal{S}'$ from the upper and lower half planes given in (2.2) be equal in $\mathcal{S}'$. Then there is an entire holomorphic function $F(z)$ such that $F(z) = f(z)$, $z \in \mathbb{R}^1 + iC$.

PROOF. First note that $O(C) = (-\infty,\infty)$. Obtain $g(t) \in \mathcal{S}'_q \subset \mathcal{S}'$, $(1/p) + (1/q) = 1$, $1 < p \leq 2$, as in Corollary 2 and define

$$F(z) = \int_{\mathbb{R}^1} g(t) \exp(2\pi i \langle z,t \rangle)\, dt = \int_{\mathbb{R}^1} g(t)\, \lambda(t) \exp(2\pi i \langle z,t \rangle)\, dt, \quad z \in \mathbb{C}^1, \tag{2.19}$$

as in (2.16). Here $(O(C))^* = \{0\}$ and $\mathrm{supp}(g) \subseteq \{t: u_{O(C)}(t) \leq A\rho_C\} = (O(C))^* + \overline{N(0;A\rho_C)} = [-A\rho_C, A\rho_C]$. Thus $g \in \mathcal{S}'_q$ has compact support here, and hence $g \in \mathcal{E}'$. $F(z)$ in (2.19) is the Fourier-Laplace transform of a distribution of compact support and hence is an entire holomorphic function in $\mathbb{C}^1$ (Hörmander [9, Theorem 1.7.5, p. 20]). $F(z) = f(z)$, $z \in \mathbb{R}^1 + iC$, as before.

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The author's permanent address is Department of Mathematics and Computer Science, Wake Forest University, Winston-Salem, North Carolina 27109, U.S.A.

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