

## LAPLACE TRANSFORM PAIRS OF N-DIMENSIONS

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ABSTRACT. In this paper I prove a theorem to obtain new n-dimensional Laplace transform pairs.

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### 1. INTRODUCTION.

The generalization of the well-known Laplace transform

$$L\{f(t) ; s\} = \int_0^{\infty} \exp(-st)f(t)dt \quad (1.1)$$

to n-dimensional Laplace transform is represented as follows:

$$\begin{aligned} L_n\{f(t_1, t_2, \dots, t_n; s_1, s_2, \dots, s_n)\} &= L_n\{f\} \\ &= \int_0^{\infty} \int_0^{\infty} \dots \int_0^{\infty} \exp\left(-\sum_{k=1}^n s_k t_k\right) f dt_1 dt_2 \dots dt_n. \end{aligned} \quad (1.2)$$

In this paper I consider a method of computing Laplace transform pairs of n-dimensions from known one-dimensional Laplace transforms. The multi-dimensional Laplace transform pairs are useful in the solution of partial differential equations (see [1], [3] and [4]).

### 2. THEOREM. Let

- (i)  $L_1\{f(t);s\} = \phi(s)$
- (ii)  $L_1\{\sqrt{t} \phi(\frac{1}{t});s\} = F(s)$
- (iii)  $L_1\{t^3 f(t^4);s\} = G(s)$
- (iv)  $L_1\{t^4 f(t^4);s\} = H(s)$

and let  $f(t), \sqrt{t} \phi(\frac{1}{t}), t^3 f(t^4), t^4 f(t^4)$  be continuous and absolutely integrable in  $(0, \infty)$ . Then

$$L_n \left\{ \frac{(\frac{1}{t_1} + \dots + \frac{1}{t_n})^3}{(t_1 \dots t_n)^{1/2}} F[\frac{1}{64}(\frac{1}{t_1} + \dots + \frac{1}{t_n})^2]; s_1, \dots, s_n \right\}$$

$$= 2^{10} \pi^{\frac{n+1}{2}} \frac{G(\sqrt{s_1} + \dots + \sqrt{s_n})}{(s_1 \dots s_n)^{1/2}} + 2^9 \pi^{\frac{n+1}{2}} \frac{\sqrt{s_1} + \dots + \sqrt{s_n}}{(s_1 \dots s_n)^{1/2}} H(\sqrt{s_1} + \dots + \sqrt{s_n}),$$

$n = 2, 3, 4, \dots$  (2.1)

provided the integral on the left exists as an absolutely convergent in each of the variables.

PROOF: From (i), we have

$$\phi(\frac{1}{s}) = \int_0^\infty e^{-t/s} f(t) dt = \int_0^\infty e^{-u/s} f(u) du,$$

$$\sqrt{t} \phi(\frac{1}{t}) = \int_0^\infty \sqrt{t} e^{-u/t} f(u) du. \tag{2.2}$$

Let us multiply both sides of (2.2) by  $e^{-st}$ ,  $\text{Re}(s) > 0$ , and integrate between the limits  $(0, \infty)$ . Then on changing the order of integrations on the resulting right hand integral (permissible by Fubini's theorem, on account of absolute convergence), we obtain

$$\int_0^\infty e^{-st} \sqrt{t} \phi(\frac{1}{t}) dt = \int_0^\infty f(u) \left[ \int_0^\infty \sqrt{t} e^{-st-u/t} dt \right] du.$$

We then evaluate the inner integral on the right (see [5], page 22) and use (ii) on the left to get the following result:

$$F(s) = \frac{\sqrt{\pi}}{2} \int_0^\infty (1 + 2\sqrt{us}) s^{-3/2} e^{-2\sqrt{us}} f(u) du,$$

$$s^{3/2} F(s) = \frac{\sqrt{\pi}}{2} \int_0^\infty (1 + 2\sqrt{us}) e^{-2\sqrt{us}} f(u) du. \tag{2.3}$$

Next let us write (2.3) in the form

$$\left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^3 F\left[\frac{1}{64}\left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^2\right]$$

$$= 256\sqrt{\pi} \int_0^\infty e^{-\frac{\sqrt{u}}{4} \sum \frac{1}{t_i}} f(u) du + 64\sqrt{\pi} \int_0^\infty \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right) e^{-\frac{\sqrt{u}}{4} \sum \frac{1}{t_i}} \sqrt{u} f(u) du$$

$$= 1024\sqrt{\pi} \int_0^\infty e^{-\frac{u^2}{4} \sum \frac{1}{t_i^2}} t_i^3 f(u^4) du + 256\sqrt{\pi} \int_0^\infty \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right) e^{-\frac{u^2}{4} \sum \frac{1}{t_i^2}} t_i^5 f(u^4) du$$

We multiply both sides by  $(t_1 \dots t_n)^{-1/2} \exp(-\sum s_i t_i)$ , integrate with respect to  $t_i$  between the limits  $(0, \infty)$  and then change the order of integrations in the resulting integral on the right, permissible by Fubini's theorem, on account of absolute convergence.

This gives

$$\begin{aligned} & \int_0^\infty \dots \int_0^\infty \exp(-\sum s_i t_i) \frac{(\frac{1}{t_1} + \dots + \frac{1}{t_n})^3}{(t_1 \dots t_n)^{1/2}} F[\frac{1}{64}(\frac{1}{t_1} + \dots + \frac{1}{t_n})^2] dt_1 \dots dt_n \\ &= 1024\sqrt{\pi} \int_0^\infty u^3 f(u^4) \left[ \int_0^\infty \frac{1}{\sqrt{t_1}} \exp(-s_1 t_1 - \frac{u^2}{4t_1}) dt_1 \dots \int_0^\infty \frac{1}{\sqrt{t_n}} \exp(-s_n t_n - \frac{u^2}{4t_n}) dt_n \right] du \\ &+ 256\sqrt{\pi} \int_0^\infty u^5 f(u^4) \left[ \int_0^\infty \dots \int_0^\infty \left( \frac{1}{t_1^{3/2} \sqrt{t_2 \dots t_n}} + \frac{1}{\sqrt{t_1} t_2^{3/2} \sqrt{t_3 \dots t_n}} + \dots + \frac{1}{\sqrt{t_1 \dots t_{n-1}} t_n^{3/2}} \right) \right. \\ &\quad \left. \cdot \exp(-s_1 t_1 - \frac{u^2}{4} \sum \frac{1}{t_i}) dt_1 \dots dt_n \right] du. \end{aligned} \tag{2.4}$$

Evaluating the inner integrals on the right by (see [5], page 22, results 6 and 7)

$$\int_0^\infty \frac{1}{\sqrt{t}} \exp(-st - \frac{u^2}{4t}) dt = \sqrt{\frac{\pi}{s}} e^{-u\sqrt{s}}, \quad \int_0^\infty t^{-3/2} \exp(-st - \frac{u^2}{4t}) dt = \frac{2\sqrt{\pi}}{u} e^{-u\sqrt{s}}$$

we get

$$\begin{aligned} & L_n \left\{ \frac{(\frac{1}{t_1} + \dots + \frac{1}{t_n})^3}{(t_1 \dots t_n)^{1/2}} F[\frac{1}{64}(\frac{1}{t_1} + \dots + \frac{1}{t_n})^2]; s_1, \dots, s_n \right\} \\ &= 1024 \pi^{\frac{n+1}{2}} (s_1 \dots s_n)^{-1/2} \int_0^\infty \exp(-u \sum \sqrt{s_i}) u^3 f(u^4) du \\ &+ 512 \pi^{\frac{n+1}{2}} \frac{(\sqrt{s_1} + \dots + \sqrt{s_n})}{(s_1 \dots s_n)^{1/2}} \int_0^\infty \exp(-u \sum \sqrt{s_i}) u^4 f(u^4) du. \end{aligned} \tag{2.5}$$

The proof is complete if we use (iii) and (iv) on the right hand side of (2.5).

3. APPLICATIONS: n-dimensional Laplace transform pairs.

Let  $f(t) = t^v$ ; so that  $L_1\{t^v; s\} = \frac{\Gamma(v+1)}{s^{v+1}} = \phi(s)$ . Then

$$L_1\{\sqrt{t} \phi(\frac{1}{t}, s)\} = L_1\{\Gamma(v+1)t^{v+3/2}; s\} = \frac{\Gamma(v+1)\Gamma(v+5/2)}{s^{v+5/2}} = F(s),$$

$$L_1\{t^3 f(t^4); s\} = L_1\{t^{4v+3}; s\} = \frac{\Gamma(4v+4)}{s^{4v+4}} = G(s),$$

$$L_1\{t^4 f(t^4); s\} = L_1\{t^{4v+4}; s\} = \frac{\Gamma(4v+4)}{s^{4v+5}} = H(s). \quad \text{Hence from (2.1), we get}$$

$$\begin{aligned} L_n\{(t_1 \dots t_n)^{-1/2} \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^{-2v-2}; s_1, \dots, s_n\} \\ = \frac{\pi^{\frac{n+1}{2}} \Gamma(4v+4)}{8^{2v-7} \Gamma(v+1) \Gamma(v+5/2)} (s_1 \dots s_n)^{-1/2} (\sqrt{s_1} + \dots + \sqrt{s_n})^{-4v-4} \\ + \frac{\pi^{\frac{n+1}{2}} \Gamma(4v+5)}{8^{2v-6} \Gamma(v+1) \Gamma(v+5/2)} (s_1 \dots s_n)^{-1/2} (\sqrt{s_1} + \dots + \sqrt{s_n})^{-4v-4}. \quad (3.1) \end{aligned}$$

Similarly if we take  $f$  to be the following

$$f(t) = \begin{cases} t^{c-1} {}_0F_3(a, b, c; kt) \\ t^v \exp(-\sqrt{t}) \\ J_v^2(\sqrt{2t}) \\ t^\alpha {}_pF_q\left(\begin{matrix} (a); \\ (b); \end{matrix} t\right) \end{cases}$$

in the theorem, then we obtain the following  $n$ -dimensional Laplace transform pairs:

$$\begin{aligned} L_n \left\{ \frac{(t_1 \dots t_n)^{-1/2}}{\left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^{2c}} {}_1F_2\left[\begin{matrix} c + 3/2; \\ a, b; \end{matrix} \frac{64k}{\left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^2}\right]; s_1, \dots, s_n \right\} \\ = \frac{\pi^{\frac{n+1}{2}} \Gamma(4c)}{8^{2c} \Gamma(c) \Gamma(c+3/2) (\sqrt{s_1} + \dots + \sqrt{s_n})^{4c}} (s_1 \dots s_n)^{-1/2} {}_4F_3 \left[ \begin{matrix} 2c, 2c+1/2, 2c+1, 2c+2; \\ a, b, c; \end{matrix} \frac{256k}{(\sqrt{s_1} + \dots + \sqrt{s_n})^4} \right] \\ + \frac{\pi^{\frac{n+1}{2}} \Gamma(4c+1)}{8^{2c} \Gamma(c) \Gamma(c+3/2) (\sqrt{s_1} + \dots + \sqrt{s_n})^{4c}} (s_1 \dots s_n)^{-1/2} {}_4F_3 \left[ \begin{matrix} 2c+1/2, 2c+1, 2c+3/2, 2c+2; \\ a, b, c; \end{matrix} \frac{256k}{(\sqrt{s_1} + \dots + \sqrt{s_n})^4} \right], \end{aligned}$$

$\text{Re}(c) > 0.$

(3.2)

$$\begin{aligned}
 & L_n \left\{ \frac{t_1^{-1} + \dots + t_n^{-1}}{(t_1 \dots t_n)^{1/2}} \left( \frac{1}{64} \left( \frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^2 - \frac{1}{4} \right)^{-v-3/2} p_1^{-2v-3} \left[ \frac{4}{\frac{1}{t_1} + \dots + \frac{1}{t_n}} \right]; s_1, \dots, s_n \right\} \\
 &= \frac{\pi^{n/2} \Gamma(4v+4)}{2^{v-4} \Gamma(2v-5)} (s_1 \dots s_n)^{-1/2} \exp\left(\frac{1}{8}(\sqrt{s_1} + \dots + \sqrt{s_n})^2\right) D_{-4v-4} \left( \frac{1}{\sqrt{2}}(\sqrt{s_1} + \dots + \sqrt{s_n}) \right) \\
 &+ \frac{\pi^{n/2} \Gamma(4v+5)}{2^{v-5} \Gamma(2v+5)} \frac{(\sqrt{s_1} + \dots + \sqrt{s_n})}{(s_1 \dots s_n)^{1/2}} \exp\left(\frac{1}{8}(\sqrt{s_1} + \dots + \sqrt{s_n})^2\right) D_{-4v-5} \left( \frac{1}{\sqrt{2}}(\sqrt{s_1} + \dots + \sqrt{s_n}) \right), \\
 & \qquad \qquad \qquad \text{Re}(v) > -1. \tag{3.3}
 \end{aligned}$$

$$\begin{aligned}
 & L_n \left\{ \frac{(t_1 \dots t_n)^{-1/2} \left( \frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^3}{\left( \frac{1}{64} \left( \frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^2 + 1 \right)^{-1}} Q_{v-1/2}^2 \left( \frac{1}{64} \left( \frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^2 + 1 \right); s_1, \dots, s_n \right\} \\
 &= \frac{\pi^{\frac{n+1}{2}} \sin(v+3/2) \pi (\sqrt{s_1} + \dots + \sqrt{s_n})^{v/2}}{2^{\frac{v}{2}-9} \sin(v-1/2) \pi (s_1 \dots s_n)^{1/2}} \\
 & \cdot G_{50}^{03} \left( \frac{8\sqrt{2}}{(\sqrt{s_1} + \dots + \sqrt{s_n})^2} \middle| 2 - \frac{v}{3}, -\frac{v}{4}, -\frac{1}{2} - \frac{v}{4}, \frac{3}{2} + \frac{v}{4}, 2 + \frac{v}{4} \right) \\
 & + \frac{\pi^{\frac{n}{2}+1} \sin(v+3/2) \pi (\sqrt{s_1} + \dots + \sqrt{s_n})^{\frac{v+3}{2}}}{s^{v-73/8} \sin(v-1/2) \pi (s_1 \dots s_n)^{1/2}} \cdot \\
 & G_{50}^{03} \left( \frac{8\sqrt{2}}{(\sqrt{s_1} + \dots + \sqrt{s_n})^2} \middle| \frac{9}{4} - \frac{v}{4}, -\frac{1}{4} - \frac{v}{4}, -\frac{3}{4} - \frac{v}{4}, \frac{7}{4} + \frac{v}{4}, \frac{9}{4} + \frac{v}{4} \right) \tag{3.4}
 \end{aligned}$$

$$\begin{aligned}
 & L_n \left\{ \frac{(t_1 \dots t_n)^{-1/2}}{\left( \frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^{2\alpha+2}} p+2 F_q \left[ \begin{matrix} (a), \alpha+1, \alpha+5/2; \\ (b); \end{matrix} \frac{64}{\left( \frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^2} \right]; s_1, \dots, s_n \right\} \\
 &= \frac{\pi^{\frac{n+1}{2}} \Gamma(4\alpha+4) (s_1 \dots s_n)^{-1/2}}{4(8)^{2\alpha+1} \Gamma(\alpha+1) \Gamma(\alpha+5/2) (\sqrt{s_1} + \dots + \sqrt{s_n})^{4\alpha+4}} \\
 & \cdot p+4 F_q \left[ \begin{matrix} (a), 2\alpha+2, 2\alpha+5/2, 2\alpha+3, 2\alpha+7/2; \\ (b); \end{matrix} \frac{256}{(\sqrt{s_1} + \dots + \sqrt{s_n})^4} \right] \\
 & + \frac{\pi^{\frac{n+1}{2}} \Gamma(4\alpha+5) (s_1 \dots s_n)^{-1/2}}{8^{2\alpha+2} \Gamma(\alpha+1) \Gamma(\alpha+5/2) (\sqrt{s_1} + \dots + \sqrt{s_n})^{4\alpha+4}} \\
 & \cdot p+4 F_q \left[ \begin{matrix} (a), 2\alpha+3, 2\alpha+7/2, 2\alpha+4, 2\alpha+9/2; \\ (b); \end{matrix} \frac{256}{(\sqrt{s_1} + \dots + \sqrt{s_n})^4} \right], \\
 & \qquad \qquad \qquad \text{Re}(\alpha) > 1. \tag{3.5}
 \end{aligned}$$

REFERENCES

- [1] Ditkin, V.A., and Prudnikov, A. P., Operational calculus in two variables and its application (English translation Pergmon Press. London (1962).
- [2] Erdelyi, A. W. Magnus, F. Oberhettinger, F. Tricomi, Tables of Integral Transforms. Vol.1, McGraw-Hill, New York (1954).
- [3] Estrin, T. A. and Higgins, T. J., Solutions of boundary value problem by multiple Laplace transformation. Jour. Frank. Inst. 252 (1951), 152-167.
- [4] Jaeger, J., The solution of boundary value problems by a double Laplace transformation. Bull. Amer. Math. Soc. 46 (1940), 687-693.
- [5] Roberts, G. E. and Kaufman, H., Table of Laplace transforms. W. B. Saunders Company (1966).



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