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RESEARCH NOTES

ON LEGENDRE NUMBERS

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ABSTRACT. The Legendre numbers, an infinite set of rational numbers, are defined from the associated Legendre functions and several elementary properties are presented. A general formula for the Legendre numbers is given. Applications include summing certain series of Legendre numbers and evaluating certain integrals. Legendre numbers are used to obtain the derivatives of all orders of the Legendre polynomials at $x = 1$.

KEY WORDS AND PHRASES. *Associated Legendre functions, Legendre polynomials, series of Legendre numbers, integrals of Legendre polynomials, orthogonal set, derivatives of Legnedre polynomials.*
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1. INTRODUCTION.

Many sets of numbers are associated with polynomials. For example, Stirling numbers of the first and second kinds, Bernoulli numbers, and Euler numbers are defined from certain polynomials. We follow this pattern by defining the Legendre numbers from the associated Legendre functions. These Legnedre numbers have many properties and applications and our purpose is to examine some of these.

2. THE LEGENDRE NUMBERS.

The associated Legnedre functions are defined by

$$P_n^m(x) = (1 - x^2)^{\frac{m}{2}} D^m(P_n(x)), \tag{2.1}$$

where $P_n(x)$ is the nth order Legendre polynomial and m and n are non-negative integers. Using Rodriques' formula, one has

$$P_n^m(x) = \frac{(1 - x^2)^{\frac{m}{2}}}{n!2^n} D^{m+n}(x^2 - 1)^n. \tag{2.2}$$

With these, the legendre numbers can be defined as follows.

Definition 1. The Legendre number, P_n^m, are the values of $P_n^m(x)$ for $x = 0$.

From (2.1) and the definition, it is clear that

$$P_n^m = P_n^{(m)}(0), \tag{2.3}$$

where $P_n^{(m)}(0)$ is the mth derivation of $P_n(x)$, evaluated at $x = 0$. From (2.2), one sees that

$$P_n^m = \frac{1}{n!2^n} D^{m+n}(x^2 - 1)^n]_{x=0}. \tag{2.4}$$

From (2.4) it is clear that $P_n^m = 0$ for $m + n$ odd and also for $m > n$. For $m + n$ even and $m \leq n$, there is exactly one term (2.4) void of x (before taking $x=0$) and this term simplifies to the third part of the explicit formula

$$P_n^m = \begin{cases} 0, & m + n \text{ odd} \\ 0, & m > n \\ \dfrac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^n(\frac{n+m}{2})!(\frac{n-m}{2})!}, & m + n \text{ even, } m \leq n . \end{cases} \tag{2.5}$$

This gives all Legendre numbers with m and n non-negative integers. Rainville, [1], gives all values of $P_n = P_n^0$ as

$$\begin{cases} P_0 & = 1 \\ P_{2n+1} & = 0 \\ P_{2n} & = \dfrac{(-1)^n(\frac{1}{2})_n}{n!} \end{cases} \tag{2.6}$$

which agrees with (2.5) for $m = 0$.

The following table gives some of the Legendre numbers. Note from (2.5) that all Legendre numbers are rational.

$n \backslash P_n^m$	$P_n = P_n^0$	P_n^1	P_n^2	P_n^3	P_n^4	P_n^5	P_n^6	P_n^7	P_n^8	...
0	1									
1	0	1								
2	$-\frac{1}{2}$	0	3							
3	0	$-\frac{3}{2}$	0	15						
4	$\frac{3}{8}$	0	$-\frac{15}{2}$	0	105					
5	0	$\frac{15}{8}$	0	$-\frac{105}{2}$	0	945				
6	$-\frac{15}{48}$	0	$\frac{105}{8}$	0	$-\frac{945}{2}$	0	10,395			
7	0	$-\frac{105}{48}$	0	$\frac{945}{8}$	0	$-\frac{10,395}{2}$	0	135,135		
8	$\frac{105}{384}$	0	$-\frac{945}{48}$	0	$\frac{10,395}{8}$	0	$-\frac{135,135}{2}$	0	2,027,025	
⋮	...									

TABLE 1. LEGENDRE NUMBERS

3. SOME BASIC PROPERTIES.

The following list of simple properties, observable from the table, can be easily proved using (2.5).

$$P_n^n = 1\cdot 3\cdot 5\cdots(2n - 1), \ n \geq 1. \tag{3.1}$$

$$P_n^m = -\frac{P_{n-1}^{m+1}}{n-m}, \ m < n, \ n \geq 1. \tag{3.2}$$

$$P_n^m = (m + n - 1)P_{n-1}^{m-1}, \ m, n \geq 1. \tag{3.3}$$

$$P_n^m = -(n - m + 2)(n + m - 1)P_n^{m-2}, \ m \geq 2. \tag{3.4}$$

$$P_n^m = -\frac{(n+m-1)}{n-m} P_{n-2}^m \ , \ n \geq 2, \ m \leq n-2. \tag{3.5}$$

$$P_n^m = (n+m-1)(n+m-3)\cdots(n-m+3)(n-m+1)P_{n-m}, \ m \geq 1. \tag{3.6}$$

Equation (3.1) gives the value on the "main" diagonal of the table. Equation (3.2) gives each entry, except the last, in a row of the table from the entry just above and to the right, while (3.3) gives each entry from the one just above and to the left. Equation (3.4) allows one to fill in the entries of a row from left to right. Equation (3.5) shows the connection between entries in the same column but two rows apart. Finally, Equation (3.6) gives each Legendre number in terms of a Legendre number in the first column of the table.

4. EXPANSIONS OF LEGENDRE POLYNOMIALS AND ASSOCIATED LEGENDRE FUNCTIONS.

By Maclaurin's expansion, the Legendre polynomials, $P_n(x)$, can be expressed as

$$P_n(x) = \sum_{m=0}^{n} \frac{P_n^{(m)}(0)x^m}{m!} . \tag{4.1}$$

Using (2.3), one has

$$P_n(x) = \sum_{m=0}^{n} \frac{P_n^m x^m}{m!} . \tag{4.2}$$

With the table, (4.2) gives a simple way of writing out P_n. For example,

$$P_5(x) = \frac{15}{8}x - \frac{105}{12}x^3 + \frac{63}{8}x^5 , \tag{4.3}$$

is easily obtained using (4.2) and the entries P_5^m, for m zero through five, from the table.

Substituting $P_n(x)$ from (4.2) into (2.1) one has

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}} D^m \sum_{m=0}^{n} \frac{P_n^m x^m}{m!} . \tag{4.4}$$

If m and n are not too large, (4.4) is easy to use to obtain $P_n^m(x)$. The table provides the summation entries and the mth derivative is then evaluated, For example,

$$\begin{aligned} P_3^2(x) &= (1-x^2)D^2 \sum_{m=0}^{3} \frac{P_3^m x^m}{m!} \qquad (4.5) \\ &= (1-x^2)(P_3^2 + P_3^3 x) \\ &= (1-x^2)(0+15x) \\ &= 15x(1-x^2). \end{aligned}$$

5. SOME SERIES AND AN INTEGRAL INVOLVING LEGENDRE NUMBERS.

Taking $x = 0$ and $t = 1$ in the known generating relation for the Legendre polynomials, see Rainville, [1],

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n , \tag{5.1}$$

gives

$$\sum_{n=0}^{\infty} P_n = 2^{-\frac{1}{2}} , \tag{5.2}$$

the sum of the non-zero terms in the first column of the table. In (5.1) take k

derivative with respect to x, then let $x = 0$ and $t = 1$ to obtain

$$\sum_{n=k}^{\infty} P_n^k = 1\cdot 3\cdot 5\cdot 7\cdots(2k-1)2^{-(\frac{2k+1}{2})} . \tag{5.3}$$

This gives the sum of the non-zero terms in the kth column of the table.

A well known series involves the Legendre numbers. Let $x = 0$ in (5.1) to obtain

$$(1+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n t^n . \tag{5.4}$$

Next, let $t = \tan\theta$ for $|\theta| < \frac{\pi}{4}$ and use the appropriate trigonometric identities to obtain

$$\begin{aligned}\cos\theta &= P_0 + P_2\tan^2\theta + P_4\tan^4\theta + P_6\tan^6\theta + \cdots \\ &= 1 - \frac{1}{2}\tan^2\theta + \frac{3}{8}\tan^4\theta - \frac{5}{16}\tan^6\theta + \cdots\end{aligned} \tag{5.5}$$

where the coefficients in the series are the Legendre numbers, P_{2n}, for $n \geq 0$.

Next an integral is evaluated. Using (4.2), one has

$$\begin{aligned}\int_0^1 P_n(x)dx &= \int_0^1 \left(\sum_{m=0}^{n} \frac{P_n^m x^m}{m!}\right)dx \\ &= \sum_{m=0}^{n} \int_0^1 \frac{P_n^m x^m}{m!}\,dx \\ &= \sum_{m=0}^{n} \left[\frac{P_n^m x^{m+1}}{(m+1)!}\right]_0^1 \\ &= \sum_{m=0}^{n} \frac{P_n^m}{(m+1)!} .\end{aligned} \tag{5.6}$$

Therefore, for n any non-negative integer, one sees that

$$\int_0^1 P_n(x)dx = \sum_{m=0}^{n} \frac{P_n^m}{(m+1)!} . \tag{5.7}$$

A better formula for this integral will be obtained in that the summation will be evaluated. Recall that the Legendre polynomials form an orthogonal set for n a positive integer. Thus,

$$\int_{-1}^1 P_n(x)dx = 0 . \tag{5.8}$$

For n positive and even,

$$\begin{aligned}\int_{-1}^1 P_n(x)dx &= \int_{-1}^0 P_n(x)dx + \int_0^1 P_n(x)dx \\ &= 2\int_0^1 P_n(x)dx,\end{aligned} \tag{5.9}$$

since $P_n(x)$ is an even function for n even. The integral and summation in (5.7) are thus seen to have the value 0 for n positive and even.

More generally,

$$\int_0^1 P_n(x)dx = \sum_{m=0}^{n} \frac{P_n^m}{(m+1)!} = -\frac{P_{n+1}}{n} , \tag{5.10}$$

for n any positive integer. Now, (5.10) certainly holds for n even since

$P_{n+1} = 0$, by (2.5). To prove (5.10) for n odd an inductive type argument, omitted here, can be used.

6. DERIVATIVE OF LEGENDRE POLYNOMIALS AT $x = 1$.

First, the 0th derivative, that is, $P_n(x)$, can be evaluated at $x = 1$. From (4.2), one has

$$P_n(1) = \sum_{m=0}^{n} \frac{P_n^m}{m!}. \tag{6.1}$$

It can be shown that $P_n(1) = 1$ by evaluating the series. An inductive type argument can be used. First, if $n = 1$, then since $P_n(x) = x$, it is clear that $P_1(1) = 1$. Also, recall that $P_0(1) = 1$. Next, if $P_k(1) = 1$, we can argue that $P_{k+1}(1) = 1$. The proof can be completed by inducting on k twice, once for k even and once for k odd. Therefore, for all positive integers n,

$$P_n(1) = \sum_{m=0}^{n} \frac{P_n^m}{m!} = 1. \tag{6.2}$$

Since $P_n(x) = 1$ for $n = 0$, (6.2) than holds for all non-negative integers n.

From (4.2), the ith derivative of $P_n(x)$ evaluated for $x = 1$ is

$$P_n^{(i)}(1) = \sum_{m=1}^{n} \frac{P_n^m}{(m-i)!}. \tag{6.3}$$

It can be shown that

$$P_n^{(i)}(1) = \frac{(n-i+1)_{2i}}{i!2^i} \tag{6.4}$$

where the numerator is in factorial notation. An inductive type proof can be used here also. The induction is on i. Since the argument is long and involved it will not be given here. With the usual agreements, $P_n^{(0)}(x) = P_n(x)$ and the factorial $k_0 = 1$ for $k \neq 0$, (6.4) holds for all non-negative integers i and n. Also, for $i = 0$, (6.4) reduces to (6.2). Equation (6.4) can easily be shown to have the forms

$$P_n^{(i)}(1) = \frac{(n+i)!}{i!(n-1)!2^i} = \frac{C(n+i,2i)}{P_i^i}. \tag{6.5}$$

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