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GENERALIZED CLASSES OF STARLIKE AND CONVEX FUNCTIONS OF ORDER α

SHIGEYOSHI OWA

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan

and

C.Y. SHEN

Department of Mathematics and Statistics
Simon Fraser University
Burnaby, B.C., V5A 1S6
Canada

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ABSTRACT. We have introduced, in this paper, the generalized classes of starlike and convex functions of order α by using the fractional calculus. We then proved some subordination theorems, argument theorems, and various results of modified Hadamard product for functions belonging to these classes. We have also established some properties about the generalized Libera operator defined on these classes of functions.

KEY WORDS AND PHRASES. *Subordination, fractional derivative, fractional integral, generalized Libera operator.*
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1. INTRODUCTION.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. Furthermore, let S denote the subclass of A consisting of all univalent functions. A function $f(z)$ of S is said to be starlike of order α if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \tag{1.2}$$

for some $\alpha(0 \leqq \alpha < 1)$ and for all $z \in U$. We use $S^*(\alpha)$ to denote the class of all starlike functions of order α. Similarly, a function $f(z)$ belonging to S is said to be convex of order α if we replace (1.2) by

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha . \tag{1.3}$$

We use $K(\alpha)$ to denote the class of all convex functions of order α. Note that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, and that $S^*(\alpha) \subset S^*(0) = S^*$, $K(\alpha) \subset K(0) = K$ and $K(\alpha) \subset S^*(\alpha)$ $(0 \leqq \alpha < 1)$. The class $S^*(\alpha)$ and $K(\alpha)$ were introduced by Robertson [1], and studied subsequently by Schild [2], MacGregor [3], Pinchuk [4], and others.

Many essentially equivalent definitions of the fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g., [5, Chapter 13], [6], [7], [8], [9], [10, p. 28 et seq.], and [11]). For our discussion, it is more convenient to use the following definitions which were employed recently by Owa [12] and by Srivastava and Owa [13].

DEFINITION 1.1. The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}}\, d\xi , \tag{1.4}$$

where $\lambda > 0$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

DEFINITION 1.2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}}\, d\xi , \tag{1.5}$$

where $0 \leqq \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed as in Definition 1.1 above.

DEFINITION 1.3. Under the hypotheses of Definition 1.2 the fractional derivative of order $n+\lambda$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) , \tag{1.6}$$

where $0 \leqq \lambda < 1$, $n \in N_0 = \{0,1,2,\ldots\}$.

Let $S^*(\alpha,\lambda)$ be the class of all functions $f(z)$ in S satisfying the inequality

$$\mathrm{Re}\left(\frac{\Lambda(\lambda,f)}{f(z)}\right) > \alpha \tag{1.7}$$

for $\lambda < 1$, $0 \leqq \alpha < 1$, and for all $z \in U$, where

$$\Lambda(\lambda,f) = \Gamma(1-\lambda) z^{1+\lambda} D_z^{1+\lambda} f(z) . \tag{1.8}$$

Also let $K(\alpha,\lambda)$ be the class of all functions $f(z)$ in S such that $\Lambda(\lambda,f) \in S^*(\alpha,\lambda)$ for $\lambda < 1$ and $0 \leqq \alpha < 1$. We note that $S^*(\alpha,0) = S^*(\alpha)$ and

$K(\alpha,0) = K(\alpha)$. Thus $S^*(\alpha,\lambda)$ and $K(\alpha,\lambda)$ are the generalizations of the classes $S^*(\alpha)$ and $K(\alpha)$, respectively. The classes $S^*(\alpha,\lambda)$ and $K(\alpha,\lambda)$ were introduced by Owa [14]. Recently, Owa and Shen [15] proved some coefficient inequalities for functions belonging to the classes $S^*(\alpha,\lambda)$ and $K(\alpha,\lambda)$.

Let T be the subclass of S consisting of all functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \tag{1.9}$$

with $a_n \geq 0$ for all n. We introduce the classes of function $T^*(\alpha,\lambda)$ and $C(\alpha,\lambda)$ as follows:

$$T^*(\alpha,\lambda) = S^*(\alpha,\lambda) \cap T \tag{1.10}$$

$$C(\alpha,\lambda) = K(\alpha,\lambda) \cap T . \tag{1.11}$$

The classes $T^*(\alpha,\lambda)$ and $C(\alpha,\lambda)$ were studied by Owa [14], and the special cases $T^*(\alpha,0)$ and $C(\alpha,0)$ were studied by Silverman [16]. Thus the classes $T^*(\alpha,\lambda)$ and $C(\alpha,\lambda)$ provide an interesting generalization of the ones considered by Silverman [16].

In sections 2, 3 and 4, we shall prove several results for functions belonging to the generalized classes $S^*(\alpha,\lambda)$, $K(\alpha,\lambda)$, $T^*(\alpha,\lambda)$, and $C(\alpha,\lambda)$. We then introduce the class $S^*(\alpha,\lambda;a,b)$ of functions in section 5. In the last section, we shall study a certain integral operator defined on A.

2. SUBORDINATION THEOREMS.

Let $f(z)$ and $g(z)$ be analytic in the unit disk U. Then we say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists a function $w(z)$ analytic in the unit disk U, which satisfies $w(0) = 0$, $|w(z)| < 1$, and $f(z) = g(w(z))$. In particular, if the function $g(z)$ is univalent in the unit disk U, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ (cf. [17], [18]).

In order to prove our first theorem, we require the following lemma due to Miller, Mocanu, and Reade [19].

LEMMA 2.1. Let $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) \geqq 0$, $f(z) \in A$, and $g(z) \in A$ with $g'(0) \neq 0$ and

$$\mathrm{Re}\left\{(\beta-1)\frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)} + 1\right\} > -\delta , \tag{2.1}$$

where

$$\delta = \min(\mathrm{Re}(\gamma), \delta_0) \tag{2.2}$$

and

$$\delta_0 = \frac{2\mathrm{Re}(\beta)\,\mathrm{Re}(\gamma)}{(|\beta+\gamma| + |\beta-\bar{\gamma}|)^2} . \tag{2.3}$$

If $f(z) \prec g(z)$, then

$$\left(\frac{1}{z^{\gamma}}\int_0^z f^{\beta}(t)t^{\gamma-1}dt\right)^{\frac{1}{\beta}} \prec \left(\frac{1}{z^{\gamma}}\int_0^z g^{\beta}(t)t^{\gamma-1}dt\right)^{\frac{1}{\beta}} . \tag{2.4}$$

Using Lemma 2.1, we can prove

THEOREM 2.1. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha,\lambda)$ $(0 \leq \alpha < 1;\ \lambda < 1)$. Then

$$\frac{1}{z}\int_0^z \left(\frac{\Lambda(\lambda,f)}{f(t)}\right) dt \prec (2\alpha-1)\left(1 + \frac{1}{z}\log(1-z)\right) , \tag{2.5}$$

where $\Lambda(\lambda,f)$ is given by (1.8).

PROOF. Note that the function $g(z)$ defined by

$$g(z) = \frac{1+(1-2\alpha)z}{1-z} , \quad (z \in U) ,$$

maps the unit disk U onto the half domain such that $\mathrm{Re}(w) > \alpha$. This implies from the definition of the class $S^*(\alpha,\lambda)$ that

$$\frac{\Lambda(\lambda,f)}{f(z)} \prec g(z) = \frac{1+(1-2\alpha)z}{1-z} . \tag{2.6}$$

Furthermore, the function $g(z)$ is analytic with $g'(0) = 2(1-\alpha) \neq 0$ and

$$1 + \mathrm{Re}\left(\frac{zg''(z)}{g'(z)}\right) > 0 , \quad (z \in U) . \tag{2.7}$$

Taking $\beta = 1$, $\gamma = 1$ in Lemma 2.1, we see that the function $g(z)$ satisfies the hypotheses of Lemma 2.1. Thus we have

$$\frac{1}{z}\int_0^z \left(\frac{\Lambda(\lambda,f)}{f(t)}\right) dt \prec \frac{1}{z}\int_0^z \frac{1+(1-2\alpha)t}{1-t}\, dt \tag{2.8}$$

which implies (2.5).

COROLLARY 2.1. Let the function $f(z)$ defined by (1.1) be in the class $S^*(0,\lambda)$ $(\lambda < 1)$. Then

$$\frac{1}{z}\int_0^z \left(\frac{\Lambda(\lambda,f)}{f(t)}\right) dt \prec -1 - \frac{1}{z}\log(1-z) . \tag{2.9}$$

COROLLARY 2.2. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha)$ $(0 \leqq \alpha < 1)$. Then

$$\frac{1}{z}\int_0^z \left(\frac{tf'(t)}{f(t)}\right) dt \prec (2\alpha-1)\left(1 + \frac{1}{z}\log(1-z)\right) . \tag{2.10}$$

Similarly, we have

THEOREM 2.2. Let the function $f(z)$ defined by (1.1) be in the class $K(\alpha,\lambda)$ $(0 \leqq \alpha < 1;\ \lambda < 1)$. Then

$$\frac{1}{z}\int_0^z \left(\frac{\Lambda(\Lambda(\lambda,f))}{\Lambda(\lambda,f)}\right) dt \prec (2\alpha-1)\left(1 + \frac{1}{z}\log(1-z)\right) , \tag{2.11}$$

where $\Lambda(\lambda,f)$ is given by (1.8).

PROOF. Note that $f(z) \in K(\alpha,\lambda)$ if and only if $\Lambda(\lambda,f) \in S^*(\alpha,\lambda)$. Consequently, on replacing $f(z)$ by $\Lambda(\lambda,f)$ in Theorem 2.1, we have Theorem 2.2.

COROLLARY 2.3. Let the function $f(z)$ defined by (1.1) be in the class

$K(0,\lambda)$ $(\lambda < 1)$. Then

$$\frac{1}{z}\int_0^z \left(\frac{\Lambda(\Lambda(\lambda,f))}{\Lambda(\lambda,f)}\right) dt \prec -1 - \frac{1}{z}\log(1-z) \ . \tag{2.12}$$

COROLLARY 2.4. Let the function $f(z)$ defined by (1.1) be in the class $K(\alpha)$ $(0 \leqq \alpha < 1)$. Then

$$\frac{1}{z}\int_0^z \left(1 + \frac{tf''(t)}{f'(t)}\right) dt \prec (2\alpha-1)\left(1 + \frac{1}{z}\log(1-z)\right) \ . \tag{2.13}$$

3. ARGUMENT THEOREMS.

In this section, we derive the argument theorems for functions belonging to the classes $S^*(\alpha,\lambda)$ and $K(\alpha,\lambda)$.

THEOREM 3.1. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha,\lambda)$ $(0 \leqq \alpha < 1;\ \lambda < 1)$. Then

$$\left|\arg\left(\frac{\Lambda_0(\lambda,f)}{f(z)}\right)\right| \leqq \sin^{-1}\left(\frac{2(1-\alpha)|z|}{1+(1-2\alpha)|z|^2}\right) \ , \quad (z \in U) \ , \tag{3.1}$$

where $\Lambda_0(\lambda,f) = z^{1+\lambda}D_z^{1+\lambda}f(z)$.

PROOF. In view of (2.6), we can write

$$\frac{\Lambda(\lambda,f)}{f(z)} = \frac{1+(1-2\alpha)w(z)}{1-w(z)} \ , \tag{3.2}$$

where $w(z)$ is an analytic function in the unit disk U and satisfies $w(0) = 0$ and $|w(z)| < 1$. We note that the linear transformation

$$p(z) = \frac{1 + Bw(z)}{1 + Aw(z)} \tag{3.3}$$

maps the disk $|w| \leqq |z|$ onto the disk.

$$\left|p(z) - \frac{1 - AB|z|^2}{1 - A^2|z|^2}\right| \leqq \frac{(B - A)|z|}{1 - A^2|z|^2} \ . \tag{3.4}$$

It follows from (3.4) that

$$\left|\frac{\Lambda(\lambda,f)}{f(z)} - \frac{1+(1-2\alpha)|z|^2}{1-|z|^2}\right| \leqq \frac{2(1-\alpha)|z|}{1-|z|^2} \ . \tag{3.5}$$

This completes the proof of Theorem 3.1.

COROLLARY 3.1. Let the function $f(z)$ defined by (1.1) be in the class $S^*(0,\lambda)$ $(\lambda < 1)$. Then

$$\left|\arg\left(\frac{\Lambda_0(\lambda,f)}{f(z)}\right)\right| \leqq \sin^{-1}\left(\frac{2|z|}{1+|z|^2}\right) \ , \quad (z \in U) \ , \tag{3.6}$$

where $\Lambda_0(\lambda,f) = z^{1+\lambda}D_z^{1+\lambda}f(z)$.

COROLLARY 3.2. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha,\lambda)$ $(0 \leqq \alpha < 1;\ \lambda < 1)$. Then

$$\frac{1-(1-2\alpha)|z|}{1+|z|} \leqq \left|\frac{\Lambda(\lambda,f)}{f(z)}\right| \leqq \frac{1+(1-2\alpha)|z|}{1-|z|}, \quad (z \in U), \tag{3.7}$$

where $\Lambda(\lambda,f)$ is given by (1.8).

PROOF. The proof is clear from (3.5).

Moreover it is easy to show that

THEOREM 3.2. Let the function $f(z)$ defined by (1.1) be in the class $K(\alpha,\lambda)$ $(0 \leqq \alpha < 1;\ \lambda < 1)$. Then

$$\left|\arg\left(\frac{\Lambda_0(\Lambda_0(\lambda,f))}{\Lambda_0(\lambda,f)}\right)\right| \leqq \sin^{-1}\left(\frac{2(1-\alpha)|z|}{1+(1-2\alpha)|z|^2}\right), \quad (z \in U), \tag{3.8}$$

where $\Lambda_0(\lambda,f) = z^{1+\lambda}D_z^{1+\lambda}f(z)$.

COROLLARY 3.3. Let the function $f(z)$ defined by (1.1) be in the class $K(0,\lambda)$ $(\lambda < 1)$. Then

$$\left|\arg\left(\frac{\Lambda_0(\Lambda_0(\lambda,f))}{\Lambda_0(\lambda,f)}\right)\right| \leqq \sin^{-1}\left(\frac{2|z|}{1+|z|^2}\right), \quad (z \in U), \tag{3.9}$$

where $\Lambda_0(\lambda,f) = z^{1+\lambda}D_z^{1+\lambda}f(z)$.

COROLLARY 3.4. Let the function $f(z)$ defined by (1.1) be in the class $K(\alpha,\lambda)$ $(0 \leqq \alpha < 1;\ \lambda < 1)$. Then

$$\frac{1-(1-2\alpha)|z|}{1+|z|} \leqq \left|\frac{\Lambda(\Lambda(\lambda,f)}{\Lambda(\lambda,f)}\right| \leqq \frac{1+(1-2\alpha)|z|}{1-|z|}, \quad (z \in U), \tag{3.10}$$

where $\Lambda(\lambda,f)$ is given by (1.8).

4. MODIFIED HADAMARD PRODUCT.

Let $f_j(z)$ $(j = 1,2)$ be defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{j,n} z^n, \qquad (a_{j,n} \geqq 0). \tag{4.1}$$

We define the modified Hadamard product $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = z - \sum_{n=2}^{\infty} a_{1,n} a_{2,n} z^n . \tag{4.2}$$

We recall here the following two lemmas due to Owa [14] before state and prove our results of this section.

LEMMA 4.1. Let the function $f(z)$ be defined by (1.9). Then $f(z)$ is in the class $T^*(\alpha,\lambda)$ $(0 \leqq \alpha < 1;\ \lambda < 1)$ if and only if

$$\sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} a_n \leqq 1 - \alpha \tag{4.3}$$

for $0 \leqq \alpha < 1$ and $\lambda < 1$. The result (4.3) is sharp.

LEMMA 4.2. Let the function $f(z)$ be defined by (1.9). Then $f(z)$ is in the class $C(\alpha,\lambda)$ $(0 \leqq \alpha < 1;\ \lambda < 1)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} a_n \leqq 1 - \alpha \tag{4.4}$$

for $0 \leqq \alpha < 1$ and $\lambda < 1$. The result (4.4) is sharp.

With the aid of Lemma 4.1, we can prove

THEOREM 4.1. Let the functions $f_j(z)$ $(j = 1,2)$ defined by (4.1) be in the class $T^*(\alpha,\lambda)$ $(0 \leqq \alpha < 1;\ \lambda < 1)$. Then the modified Hadamard product $f_1*f_2(z)$ is in the class $T^*(\beta(\alpha,\lambda),\lambda)$, where

$$\beta(\alpha,\lambda) = \frac{1 - \dfrac{2(1-\alpha)^2(1-\lambda)}{(2-\alpha+\alpha\lambda)^2}}{1 - \dfrac{(1-\alpha)^2(1-\lambda)^2}{(2-\alpha+\alpha\lambda)^2}} . \tag{4.5}$$

The result is sharp.

PROOF. We use a technique due to Schild and Silverman [20]. It is sufficient to prove that

$$\sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \beta\right\} a_{1,n}a_{2,n} \leqq 1 - \beta \tag{4.6}$$

for $\beta \leqq \beta(\alpha,\lambda)$. By using the Cauchy-Schwarz inequality, we know from (4.3) that

$$\sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} \sqrt{a_{1,n}a_{2,n}} \leqq 1-\alpha . \tag{4.7}$$

Therefore we need find the largest β such that

$$\sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \beta\right\} a_{1,n}a_{2,n}/(1-\beta)$$

$$\leqq \sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} \sqrt{a_{1,n}a_{2,n}}/(1-\alpha), \tag{4.8}$$

or

$$\sqrt{a_{1,n}a_{2,n}} \leqq \frac{(1-\beta)\left\{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}}{(1-\alpha)\left\{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \beta\right\}} \quad (n \geqq 1) . \tag{4.9}$$

In view of (4.7), we observe that it suffices to find the largest β such that

$$\frac{1-\alpha}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha} \leqq \frac{(1-\beta)\left\{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}}{(1-\alpha)\left\{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \beta\right\}} . \tag{4.10}$$

We note that (4.10) gives

$$\beta \leqq \frac{1 - H(n)\left(\dfrac{1-\alpha}{H(n)-\alpha}\right)^2}{1 - \left(\dfrac{1-\alpha}{H(n)-\alpha}\right)^2} \qquad (n \geqq 1) , \tag{4.11}$$

where

$$H(n) = \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} . \tag{4.12}$$

Since, for fixed α ,

$$\frac{1 - H(n)\left(\frac{1-\alpha}{H(n)-\alpha}\right)^2}{1-\left(\frac{1-\alpha}{H(n)-\alpha}\right)^2}$$

is an increasing function of n, we have

$$\beta \leqq \frac{1 - H(2)\left(\frac{1-\alpha}{H(2)-\alpha}\right)^2}{1-\left(\frac{1-\alpha}{H(2)-\alpha}\right)^2} = \frac{1-\dfrac{2(1-\alpha)^2(1-\lambda)}{(2-\alpha+\alpha\lambda)^2}}{1-\dfrac{(1-\alpha)^2(1-\lambda)^2}{(2-\alpha+\alpha\lambda)^2}}. \tag{4.13}$$

Finally, by taking the functions $f_j(z)$ $(j = 1,2)$ defined by

$$f_j(z) = z - \frac{(1-\alpha)(1-\lambda)}{2-\alpha+\alpha\lambda} z^2 , \tag{4.14}$$

we can see that the result is sharp.

COROLLARY 4.1. Let the functions $f_j(z)$ $(j = 1,2)$ defined by (4.1) be in the class $T^*(0,\lambda)$ $(\lambda < 1)$. Then the modified Hadamard product $f_1*f_2(z)$ is in the class $T^*(\beta(\lambda),\lambda)$, where

$$\beta(\lambda) = \frac{2(1+\lambda)}{3+2\lambda-\lambda^2} . \tag{4.15}$$

The result is sharp.

By using the same technique and Lemma 4.2, we have

THEOREM 4.2. Let the functions $f_j(z)$ $(j = 1,2)$ defined by (4.1) be in the class $C(\alpha,\lambda)$ $(0 \leqq \alpha < 1;\ \lambda < 1)$. Then the modified Hadamard product $f_1*f_2(z)$ is in the class $C(\gamma(\alpha,\lambda),\lambda)$ where

$$\gamma(\alpha,\lambda) = \frac{1-\dfrac{(1-\alpha)^2(1-\lambda)^2}{(2-\alpha+\alpha\lambda)^2}}{1-\dfrac{(1-\alpha)^2(1-\lambda)^3}{2(2-\alpha+\alpha\lambda)^2}}. \tag{4.16}$$

The result is sharp for the functions $f_j(z)$ $(j = 1,2)$ defined by

$$f_j(z) = z - \frac{(1-\alpha)(1-\lambda)^2}{2(2-\alpha+\alpha\lambda)} z^2 . \tag{4.17}$$

COROLLARY 4.2. Let the functions $f_j(z)$ $(j = 1,2)$ defined by (4.1) be in the class $C(0,\lambda)$ $(\lambda < 1)$. Then the modified Hadamard product $f_1*f_2(z)$ is in the class $C(\gamma(\lambda),\lambda)$, where

$$\gamma(\lambda) = \frac{2(3+2\lambda-\lambda^2)}{7+3\lambda-3\lambda^2+\lambda^3} . \tag{4.18}$$

The result is sharp.

5. THE CLASS $S^*(\alpha,\lambda;a,b)$.

Let the function $f(z)$ defined by (1.1) be in the class A and let

$$P(f(z);\alpha,\lambda;a,b) = \left(\frac{\Lambda(\lambda,f)}{f(z)} - \alpha\right)^a \left(\frac{\Lambda(\Lambda(\lambda,f)}{\Lambda(\lambda,f)} - \alpha\right)^b , \tag{5.1}$$

where a and b are real numbers. Then we say that $f(z)$ is in the class $S^*(\alpha,\lambda;a,b)$ if $f(z)$ satisfies

$$\mathrm{Re}\{P(f(z);\alpha,\lambda;a,b)\} > 0 \tag{5.2}$$

for $0 \leqq \alpha < 1$, $\lambda < 1$ and $z \in U$.

We observe that $S^*(\alpha,\lambda;1,0) = S^*(\alpha,\lambda)$ and $S^*(\alpha,\lambda;0,1) = K(\alpha,\lambda)$.

THEOREM 5.1. Let $0 \leqq \alpha < 1$, $\lambda < 1$, and $0 \leqq t \leqq 1$. Then

$$S^*(\alpha,\lambda;a,b) \cap S^*(\alpha,\lambda;1,0) \subset S^*(\alpha,\lambda;(a-1)t+1,bt). \tag{5.3}$$

PROOF. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha,\lambda;a,b) \cap S^*(\alpha,\lambda;1,0)$ and let

$$V(z) = \left(\frac{\Lambda(\lambda,f)}{f(z)} - \alpha\right)^{a}\left(\frac{\Lambda(\Lambda(\lambda,f))}{\Lambda(\lambda,f)} - \alpha\right)^{b}. \tag{5.4}$$

Then we have $\mathrm{Re}(V(z)) > 0$ for $z \in U$. Further, setting

$$U(z) = \frac{\Lambda(\lambda,f)}{f(z)} - \alpha, \tag{5.5}$$

we have $\mathrm{Re}(U(z)) > 0$ for $z \in U$. It follows from (5.4) and (5.5) that

$$\left(\frac{\Lambda(\lambda,f)}{f(z)} - \alpha\right)^{(a-1)t+1}\left(\frac{\Lambda(\Lambda(\lambda,f))}{\Lambda(\lambda,f)} - \alpha\right)^{bt} = (U(z))^{1-t}(V(z))^{t}. \tag{5.6}$$

Define the function $F(z)$ by

$$F(z) = (U(z))^{1-t}(V(z))^{t}. \tag{5.7}$$

Then we have $F(0) = (1-\alpha)^{(a+b-1)t+1} > 0$ and

$$|\arg(F(z))| \leqq (1-t)|\arg(U(z))| + t|\arg(V(z))| \leqq \frac{\pi}{2}, \tag{5.8}$$

which shows that $F(z)$ maps the unit disk U onto a domain which is contained in the right half-plane. Hence we complete the proof of Theorem 5.1.

COROLLARY 5.1. Let $0 \leqq \alpha < 1$, $\lambda < 1$, and $0 \leqq t \leqq 1$. Then

$$S^*(\alpha,\lambda;a,b) \cap S^*(\alpha,\lambda;0,1) \subset S^*(\alpha,\lambda;\, at,\, (b-1)t+1). \tag{5.9}$$

6. GENERALIZED LIBERA OPERATOR $J_c(f)$.

For a function $f(z)$ belonging to the class A, we define the operator $J_c(f)$ by

$$J_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt \qquad (c \geqq 0). \tag{6.1}$$

This operator $J_c(f)$ when c is a natural number was studied by Bernardi [21]. In particular, the operator $J_1(f)$ was studied by Libera [22], Livingston [23], and Mocanu, Reade and Ripeanu [24]. It follows from (6.1) that

$$J_c(f) = \frac{c+1}{z^c}\int_0^z (\sum_{n=1}^{\infty} a_n t^{n+c-1})dt \qquad (a_1 = 1)$$

$$= \sum_{n=1}^{\infty} (\frac{c+1}{n+c}) a_n z^n . \tag{6.2}$$

In order to prove our theorem, we recall here the following theorem due to Jack [25].

LEMMA 6.1. Let $w(z)$ be regular in the unit disk U, with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r$ $(0 \leqq r < 1)$ at a point z_0, we can write

$$z_0 w'(z_0) = mw(z_0) , \tag{6.3}$$

where m is real and $m \geqq 1$.

Furthermore, we need the following lemma by Pascu [26].

LEMMA 6.2. If $f(z) \in S^*$, then $J_c(f) \in S^*$.

With the aid of Lemmas 6.1 and 6.2, we prove

THEOREM 6.1. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha,\lambda) \cap S^*$ $(0 \leqq \alpha < 1;\ \lambda < 1)$. Then the functional $J_c(f)$ is in the class $S^*(\alpha,\lambda)$.

PROOF. Define the function $w(z)$ by

$$\frac{\Lambda(\lambda,J_c(f))}{J_c(f)} = \frac{1+(1-2\alpha)w(z)}{1-w(z)} \qquad (w(z) \neq 1) . \tag{6.4}$$

Then $w(z)$ is a regular function in the unit disk U with $w(0) = 0$. Differentiating both sides of (6.4), we obtain

$$\frac{z(\Lambda(\lambda,J_c(f)))'}{J_c(f)} - \frac{z(J_c(f))'}{J_c(f)} \cdot \frac{\Lambda(\lambda,J_c(f))}{J_c(f)} = \frac{2(1-\alpha)zw'(z)}{(1-w(z))^2} . \tag{6.5}$$

Note that

$$z(J_c(f))' = z(\sum_{n=1}^{\infty} (\frac{c+1}{n+c})\ a_n z^n)' \qquad (a_n = 1)$$

$$= \sum_{n=1}^{\infty} \frac{n(c+1)}{n+c}\ a_n z^n$$

$$= \sum_{n=1}^{\infty} \{(c+1) - \frac{c(c+1)}{n+c}\}\ a_n z^n$$

$$= (c+1)f(z) - c\ J_c(f, \tag{6.6}$$

and $z(\Lambda(\lambda,J_c(f)))' = z[\Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}J_c(f)]'$

$$= z[\Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}(\sum_{n=1}^{\infty} (\frac{c+1}{n+c})\ a_n z^n)]'$$

$$= z(\sum_{n=1}^{\infty} \frac{(c+1)\Gamma(n+1)\Gamma(1-\lambda)}{(n+c)\Gamma(n-\lambda)}\ a_n z^n)'$$

$$= \sum_{n=1}^{\infty} \frac{n(c+1)\Gamma(n+1)\Gamma(1-\lambda)}{(n+c)\Gamma(n-\lambda)} a_n z^n$$

$$= \sum_{n=1}^{\infty} \{(c+1) - \frac{c(c+1)}{n+c}\} \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} a_n z^n$$

$$= (c+1) \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} a_n z^n - c \sum_{n=1}^{\infty} \frac{(c+1)\Gamma(n+1)\Gamma(1-\lambda)}{(n+c)\Gamma(n-\lambda)} a_n z^n$$

$$= (c+1)\Lambda(\lambda,f) - c\Lambda(\lambda,J_c(f)) . \tag{6.7}$$

Applying (6.6) and (6.7) to (6.5) we have

$$(c+1) \frac{\Lambda(\lambda,f)}{J_c(f)} - (c+1) \frac{f(z)}{J_c(f)} \cdot \frac{\Lambda(\lambda,J_c(f))}{J_c(f)} = \frac{2(1-\alpha)zw'(z)}{(1-w(z))^2} \tag{6.8}$$

or

$$\frac{\Lambda(\lambda,f)}{f(z)} = \frac{\Lambda(\lambda,J_c(f))}{J_c(f)} + \frac{2(1-\alpha)zw'(z)}{(1-w(z))^2} \cdot \frac{J_c(f)}{(c+1)f(z)} . \tag{6.9}$$

Since

$$\frac{(c+1)f(z)}{J_c(f)} = \frac{z(J_c(f))'}{J_c(f)} + c , \tag{6.10}$$

equation (6.9) becomes

$$\frac{\Lambda(\lambda,f)}{f(z)} = \frac{1+(1-2\alpha)w(z)}{1-w(z)} + \frac{2(1-\alpha)zw'(z)}{(1-w(z))^2} \cdot \frac{1}{\frac{z(J_c(f))'}{J_c(f)} + c} . \tag{6.11}$$

Assume that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq 1) . \tag{6.12}$$

Then, Lemma 6.1. implies that

$$z_0 w'(z_0) = m\, w(z_0) ,$$

where m is real and $m \geqq 1$. Applying Lemma 6.1 to $w(z)$, and putting $w(z_0) = e^{i\theta_0}$, we know that

$$\mathrm{Re}\left(\frac{\Lambda(\lambda,f)}{f(z)}\right) = \mathrm{Re}\left\{\frac{1+(1-2\alpha)w(z_0)}{1-w(z_0)}\right\} + \mathrm{Re}\left\{\frac{2(1-\alpha)mw(z_0)}{(1-w(z_0))^2} \cdot \frac{1}{\frac{z_0(J_c(f))'}{J_c(f)} + c}\right\}$$

$$= \alpha - \frac{(1-\alpha)m}{1-\cos\theta_0} \mathrm{Re}\left\{\frac{1}{\frac{z_0(J_c(f))'}{J_c(f)} + c}\right\} .$$

We note that Lemma 6.2 gives, for $f(z) \in S^*$,

$$\operatorname{Re}\left\{\frac{1}{\dfrac{z_0(J_c(f))'}{J_c(f)}+c}\right\}$$

$$= \frac{\operatorname{Re}\left(\dfrac{z_0(J_c(f))'}{J_c(f)}\right)+c}{\left\{\operatorname{Re}\left(\dfrac{z_0(J_c(f))'}{J_c(f)}\right)+c\right\}^2+\left\{I_m\left(\dfrac{z_0(J_c(f))'}{J_c(f)}\right)\right\}^2}$$

$$> 0 \quad . \tag{6.14}$$

Consequently, we conclude that

$$\operatorname{Re}\left(\frac{\Lambda(\lambda,f)}{f(z)}\right) \leqq \alpha \tag{6.15}$$

which contradicts the condition $f(z) \in S^*(\alpha,\lambda)$.

Thus we have $|w(z)| < 1$ for all $z \in U$. This proves from (6.4) that

$$\operatorname{Re}\left(\frac{\Lambda(\lambda,J_c(f))}{J_c(f)}\right) = \operatorname{Re}\left(\frac{1+(1-2\alpha)w(z)}{1-w(z)}\right) > \alpha ,$$

or, that $J_c(f) \in S^*(\alpha,\lambda)$. Hence we have completed the proof of the theorem.

COROLLARY 6.1. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha)$ $(0 \leqq \alpha < 1)$. Then the operator $J_c(f)$ is also in the class $S^*(\alpha)$.

PROOF. Taking $\lambda = 0$ in Theorem 6.1, we have

$$f(z) \in S^*(\alpha) \cap S^* \Rightarrow J_c(f) \in S^*(\alpha) .$$

Noting $S^*(\alpha) \subset S^*$ for $0 \leqq \alpha < 1$, we have the corollary.

Finally, we state and prove

THEOREM 6.2. Let the function $f(z)$ defined by (1.1) be in the class $K(\alpha,\lambda)$ with $\alpha(0 \leqq \alpha < 1)$ and $\lambda(\lambda < 1)$ such that $S^*(\alpha,\lambda) \subset S^*$. Then the operator $J_c(f)$ is also in the class $K(\alpha,\lambda)$.

PROOF. In view of Theorem 8, we observe that $J_c(f) \in S^*(\alpha,\lambda)$ for $f(z) \in S^*(\alpha,\lambda) \subset S^*$. From the definition of the class $K(\alpha,\lambda)$, we obtain that

$$\begin{aligned} f(z) \in K(\alpha,\lambda) &\Rightarrow \Lambda(\lambda,f) \in S^*(\alpha,\lambda) \\ &\Rightarrow J_c(\Lambda(\lambda,f)) \in S^*(\alpha,\lambda) \\ &\Rightarrow \Lambda(\lambda,J_c(f)) \in S^*(\alpha,\lambda) \\ &\Rightarrow J_c(f) \in K(\alpha,\lambda) \ . \end{aligned}$$

This completes the proof of the theorem.

COROLLARY 6.2. Let the definition $f(z)$ defined by (1.1) be in the class $K(\alpha)$ $(0 \leqq \alpha < 1)$. Then the operator $J_c(f)$ is also in the class $K(\alpha)$.

PROOF. Letting $\lambda = 0$ in Theorem 6.2, we can easily obtain the result.

REFERENCES

1. ROBERTSON, M.S. On the theory of univalent functions, Ann. of Math. 37 (1936), 374-408.

2. SCHILD, A., On starlike functions of order α, Amer. J. Math. 87 (1965), 65-70.

3. MacGREGOR, T.H., The radius of convexity for starlike functions of order $\frac{1}{2}$, Proc. Amer. Math. Soc. 14 (1963), 71-76.

4. PINCHUK, B. On starlike and convex functions of order α, Duke Math. J. 35 (1968), 721-734.

5. ERDELYI, A., MAGNUS, W., OBERHETTINGER, F. and TRICOMI, F.G., Tables of Integral Transforms, Vol. II, McGraw-Hill Book Co., New York, London and Toronto, 1954.

6. NISHIMOTO, K. Fractional derivative and integral, Part I, J. College Engrg. Nihon University, Ser. B, 17 (1976), 11-19.

7. OSLER, T.J. Leibniz rule for fractional derivatives generalized and an application to infinite series, SIAM J. Appl. Math. 18 (1970), 658-674.

8. ROSS, B. A brief history and exposition of the fundamental theory of fractional calculus, in Fractional Calculus and Its Applications (B. Ross, Ed.), pp. 1-36, Springer-Verlag, Berlin, Heidelberg and New York, 1975.

9. SAIGO, M., A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep. College General Ed. kyushu Univ. 11 (1978), 135-143.

10. SRIVASTAVA, H.M. and BUSCHMAN, R.G., Convolution Integral Equations with Special Function Kernels, John Wiley and Sons, New York, London, Sydney and Toronto, 1977.

11. SRIVASTAVA, H.M., OWA, S. and NISHIMOTO, K. Some fractional differintegral equations, J. Math. Anal. Appl. 106 (1985), 360-366.

12. OWA, S. On the distortion theorems. I, kyungpook Math. J. 18 (1978), 53-59.

13. SRIVASTAVA, H.M. and OWA, S., An application of the fractional derivative, Math. Japan, 29 (1984), 838-389.

14. OWA, S. On new subclasses of analytic and univalent functions, to appear.

15. OWA, S. and SHEN, C.Y. On the coefficients of generalized starlike or convex functions of order α, to appear.

16. SILVERMAN, H. Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51 (1975), 109-116.

17. POMMERENKE, Ch. Univalent Functions, Vandenhoeck and Ruprecht, Göttingen, 1975.

18. DUREN, P.L. Univalent Functions, Grundleheren der Mathematischen Wissenschaften 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

19. MILLER, S.S., MOCANU, P.T. and READE, M.O. Subordination preserving integral operators, Prepr. Babes-Bolyai Univ. Fac. Math. Res. Semin. 4 (1983), 42-56.

20. SCHILD, A. and SILVERMAN, H. Convolutions of univalent functions with negative coefficients, Ann. Univ. Mariae Curie-Sklodowska, 29 (1975), 99-107.

21. BERNARDI, S.D. Convex and starlike univalent functions, Trans. Amer. Math. Soc., 135 (1969), 429-446.

22. LIBERA, R.J. Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16 (1965), 755-758.

23. LIVINGSTON, A.E. On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 17 (1966), 352-357.

24. MOCANU, P.T., READE, M.O. and RIPEANU, D. The order of starlikeness of a Libera integral operator, Mathematica 16 (1977), 67-73.

25. JACK, I.S. Functions starlike and convex of order α, J. London Math. Soc. (2) 3 (1971), 469-474.

26. PASCU, N.N. Contributii la teoria reprezentarilor conforme, teza de doctorat, Cluj-Napoca, 1978.