INCLUSIONS OF HARDY ORLICZ SPACES

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ABSTRACT. Let ϕ be a continuous positive increasing function defined on $[0, \infty)$ such that $\phi(x + y) \leq \phi(x) + \phi(y)$ and $\phi(0) = 0$. The Hardy-Orlicz space generated by ϕ is denoted by $H(\phi)$. In this paper, we prove that for $\phi \neq \psi$, if $H(\phi) = H(\psi)$ as sets, then $H(\phi) = H(\psi)$ as topological vector spaces. Some other results are given.

KEY WORDS AND PHRASES. Modulus function, Orlicz spaces.

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1. INTRODUCTION.

Let $\phi: [0, \infty) \longrightarrow [0, \infty)$ be a continuous increasing function such that $\phi(x + y) \leq \phi(x) + \phi(y)$ and $\phi(0) = 0$. Let T be the unit circle, and m be the Lebesgue measure on T . A complex valued measurable function f defined on T is called ϕ -integrable if $\int_1 \phi|f(t)| dm(t) < \infty$. The space of all ϕ -integrable functions on T will be denoted by $L(\phi)$. This space was first introduced by Orlicz, [8]. Subsequent papers were written to study different aspects of $L(\phi)$. Examples of these papers are Cater, [4], Gramsch, [5] and Pallashke [9].

In [6] and [7], Lesniewicz introduced the so called Hardy-Orlicz spaces $H(\phi)$ for a given such function ϕ . The space $H(\phi)$ was defined to be the space of all functions $f \in L(\phi)$ such that f is the radial limit of some function g analytic in the open unit disc and belongs to the Nevalinna class N . The relation between different $H(\phi)$ -spaces was studied by Deeb, Khalil and Marzug [3]. In this paper, we show that the inclusion map between two $H(\phi)$ -spaces is always continuous. Some other results are given. It should be remarked that in the work of Lesniewicz, [6], [7] and many other authors, ϕ is assumed to be a ϕ -convex function. In this paper it is not assumed so.

2. PRELIMINARIES AND NOTATIONS.

A function $\phi: [0, \infty) \longrightarrow [0, \infty)$ is called a modulus function if

- (i) ϕ is continuous and increasing
- (ii) $\phi(x) = 0$ if and only if $x = 0$
- (iii) $\phi(x + y) \leq \phi(x) + \phi(y)$.

The functions $\phi(x) = x^p$, $0 < p \leq 1$ and $\phi(x) = \ln(1 + x)$ are examples of modulus functions. Further, if ϕ_1 and ϕ_2 are modulus functions, then $\phi_1 + \phi_2$ and $\phi_1 \circ \phi_2$

are modulus functions. Further, $\psi = \frac{\phi}{1+\phi}$ is a modulus function if ϕ is.

Let $T = \{z: |z| = 1\}$, $\Delta = \{z: |z| < 1\}$. The space of analytic functions on Δ is denoted by $H(\Delta)$. Let $H^+(\Delta) = \{f \in H(\Delta): \lim_{r \rightarrow 1} f(re^{i\theta}) \text{ exists a.e. } \theta\}$. We will consider $H^+(\Delta)$ as a space of functions on T . For a given modulus function ϕ , we define:

$$H(\phi) = \{f \in H^+(\Delta): \sup_{0 \leq r < 1} \int_0^{2\pi} \phi |f(re^{i\theta})| d\theta = \int_0^{2\pi} \phi |f(e^{i\theta})| d\theta < \infty\}.$$

The function $d: H(\phi) \times H(\phi) \longrightarrow [0, \infty)$, $d(f, g) = \int_0^{2\pi} \phi |f(e^{i\theta}) - g(e^{i\theta})| d\theta$ defines a metric on $H(\phi)$, under which $H(\phi)$ becomes a topological vector space. If one assumes that $\phi|u|$ is subharmonic for $u \in H(\Delta)$, then $H(\phi)$ turns out to be complete [2]. For $f \in H(\phi)$, we write $\|f\|_\phi = \int_T \phi |f(e^{i\theta})| d\theta$. If $\phi(x) = x^p$, $0 < p \leq 1$, then $H(\phi) = H^p$ and for $\phi(x) = \ln(1+x)$, $H(\phi) = N^+ = \{f \in N: \int_T \ln(1+|f|) < \infty\}$, where N is the Nevalinna class.

3. $I: H^1 \longrightarrow H(\phi)$ IS CONTINUOUS.

In [2], it was shown that $H^1 \subseteq H(\phi)$ for all modulus functions ϕ . The authors in [3] were not able to show that the inclusion map $I: H^1 \longrightarrow H(\phi)$ is continuous. In this section we prove that $I: H^1 \longrightarrow H(\phi)$ is continuous. Some other related questions are discussed.

THEOREM 2.1. Let ϕ and ψ be two modulus functions such that $\lim_{x \rightarrow \infty} \frac{\phi(x)}{\psi(x)} = \lambda$ exists. Then:

- (i) $H(\phi) = H(\psi)$ if $\lambda \neq 0$ and λ is finite
- (ii) $H(\phi) \subseteq H(\psi)$ if $\lambda = 0$
- (iii) $H(\psi) \subseteq H(\phi)$ if $\lambda = \infty$.

PROOF. (i) Let $\lambda \neq 0$ be finite. Then there exists $a_1, b_1, a_2, b_2 \in [0, \infty)$ such that

$$\phi(x) \leq a_1 \psi(x) \text{ for } x \in [a_2, \infty) \dots (*)$$

$$\psi(x) \leq b_1 \phi(x) \text{ for } x \in [b_2, \infty) \dots (**).$$

Let $f \in H(\psi)$. Set $E(a_2) = \{t \in T: |f(t)| \geq a_2\}$. Then

$$\|f\|_\phi = \int_{E(a_2)} \phi |f(e^{i\theta})| d\theta + \int_{E^c(a_2)} \phi |f(e^{i\theta})| d\theta$$

$$\leq a_1 \|f\|_\psi + \phi(a_2) < \infty.$$

Hence $f \in H(\phi)$ and $H(\psi) \subseteq H(\phi)$. Similarly we show $H(\phi) \subseteq H(\psi)$. Consequently, $H(\phi) = H(\psi)$. Case (ii) and (iii) are proved similarly and details are omitted. This ends the proof.

THEOREM 2.2. Let $\lim_{x \rightarrow \infty} \frac{\phi(x)}{\psi(x)} = \lambda > 0$. Then the inclusion map $I: H(\phi) \longrightarrow H(\psi)$ is continuous.

PROOF. From the proof of Theorem 2.1, there exists $a, b > 0$ such that $\|f\|_\psi \leq \psi(a) + b \|f\|_\phi$ for all $f \in H(\phi)$.

Let $f_n \rightarrow 0$ in $H(\phi)$. Thus the sequence (f_n) is bounded in the metric of $H(\phi)$ and consequently bounded in $H(\psi)$. If possible let there exist a subsequence (f_{n_k})

such that $\|f_{n_k}\| \rightarrow \alpha > 0$. Since $\|f_{n_k}\|_\phi \rightarrow 0$, (f_{n_k}) has a subsequence which converges pointwise to the zero function. With no loss of generality, we can assume that $f_{n_k} \rightarrow 0$ a.e. Another application of the proof of Theorem 2.1, yields $\psi(x) \leq \psi(a) + b \cdot \phi|x|$ for all $x \in [0, \infty)$. Hence

$$\psi |f_{n_k}(t)| \leq \psi(a) + b \cdot \phi |f_{n_k}(t)|.$$

The sequence of functions $g_{n_k} = \psi(a) + b \cdot \phi |f_{n_k}|$ converges a.e. to $\psi(a)$ and

$$\int_T g_{n_k}(t) dt \rightarrow \psi(a).$$

Consequently, by the generalized Lebesgue convergence theorem, [10], we have

$$\lim_{n_k} \int_T \psi |f_{n_k}(t)| dt = \int_T \lim_{n_k} \psi |f_{n_k}(t)| dt = 0.$$

This is a contradiction. Thus, the point $w = 0$ is the only limit point of the bounded sequence $(\|f_n\|_\psi)$. Consequently, [11], the sequence $\|f_n\|_\psi$ converges to zero. Hence $I: H(\phi) \rightarrow H(\psi)$ is continuous. This ends the proof.

COROLLARY 2.3. If $\lim_{x \rightarrow \infty} \frac{\phi(x)}{\psi(x)} = \lambda \in (0, \infty)$, then $H(\phi) = H(\psi)$ as topological vector spaces.

PROOF. By Theorem 2.1, $H(\phi) = H(\psi)$ as sets. Theorem 2.2 implies that $I: H(\phi) \rightarrow H(\psi)$ is an isomorphism. This ends the proof.

A linear map $A: H(\phi) \rightarrow H(\psi)$ is called metrically bounded if $\|Af\|_\psi \leq \lambda \|f\|_\phi$ for all $f \in H(\phi)$ and some $\lambda > 0$. Clearly every metrically bounded map is continuous. The converse need not be true. However, for the inclusion map, we have the following:

THEOREM 2.4. Let ϕ be any modulus function. Then there exists $\lambda > 0$ such that for all $f \in H^1$, $\|f\|_1 \geq 1$, $\|f\|_\phi \leq \lambda \|f\|_1$.

PROOF. It is known, [2] (and easy to show) that $H^1 \subseteq H(\phi)$ for all modulus functions ϕ . If $f \in H^1$ and $\|f\|_1 = 1$, then using the argument in Theorem 2.1, we have $\|f\|_\phi \leq \lambda \|f\|_1$.

Let $f \in H^1$, $\|f\|_1 > 1$. Then there exists $0 < \alpha < 1$ such that $\|\alpha f\|_1 = 1$. Since $\alpha < 1$, there exists a natural number n such that $\frac{1}{n+1} \leq \alpha \leq \frac{1}{n}$. Hence

$$\|\alpha f\|_\phi \leq \lambda \|\alpha f\|_1 = \lambda \alpha \|f\|_1.$$

But $\|\frac{1}{n+1} f\|_\phi \leq \|\alpha f\|_\phi$, and $\|\frac{1}{k} f\|_\phi \geq \frac{1}{k} \|f\|_\phi$ for any modulus function ϕ . It follows that:

$$\frac{1}{n+1} \|f\|_\phi \leq \lambda \cdot \alpha \|f\|_1 \leq \frac{\lambda}{n} \|f\|_1,$$

and consequently

$$\|f\|_\phi \leq \lambda \frac{n+1}{n} \|f\|_1 \leq 2\lambda \|f\|_1.$$

This ends the proof.

THEOREM 2.5. Let ϕ be a given modulus function such that $H^1 = H(\phi)$. If metric and topological bounded sets coincide in $H(\phi)$, then $\|f\|_1 \leq \lambda \|f\|_\lambda$ for all $f \in H(\phi)$, $\|f\|_\phi \leq 1$ for some $\lambda > 0$.

PROOF. Applying Corollary 2.3, I: $H(\phi) \rightarrow H^1$ is an isomorphism of topological vector spaces. If possible, let $\|f\|_1 \leq \lambda \|f\|_\phi$ be not true on the unit sphere of $H(\phi)$. Then, for each n , there exists $f_n \in H(\phi)$, $\|f_n\|_\phi = 1$ such that

$$\|f_n\|_1 \geq n \|f_n\|_\phi = n$$

Consider the sequence $\frac{f_n}{n} = g_n$. By the assumption on bounded sets of $H(\phi)$, we have, [12], $g_n \rightarrow 0$ in $H(\phi)$. But $\|g_n\|_1 = \|\frac{f_n}{n}\|_1 \geq 1$ for all n . This contradicts the continuity of the identity map $I: H(\phi) \rightarrow H^1$. Hence there exists $\lambda > 0$ such that:

$$\|f\|_1 \leq \lambda \|f\|_\phi \quad \dots \quad (*),$$

for all $f \in H(\phi)$, $\|f\|_\phi = 1$.

Let $f \in H(\phi)$, $\|f\|_\phi < 1$. Consider the map $K: [0, \infty) \rightarrow [0, \infty)$, $K(t) = \|tf\|_\phi$. It can be easily seen that K is continuous. Hence there exists $a > 1$ such that $K(a) = 1$. Thus for every $f \in H(\phi)$, $\|f\|_\phi < 1$, we can find $a > 1$ such that $\|af\|_\phi = 1$. Hence, from equation (*), we get:

$$\|af\|_1 \leq \lambda \|af\|_\phi \leq 2a\lambda \|f\|_\phi.$$

Consequently, $\|f\|_1 \leq 2\lambda \|f\|_\phi$. This ends the proof.

4. FURTHER RESULTS

The concept of metrically bounded linear operator was introduced in Section 3. A linear map $A: H(\phi) \rightarrow H(\psi)$ is called metrically bounded if there exists $\lambda \in (0, \infty)$ such that $\|Af\|_\psi \leq \lambda \|f\|_\phi$. In general, a continuous linear map need not be metrically bounded. In this section we prove a result which is a generalization of Theorem 3.1 in [3].

THEOREM 4.1. Let ϕ and ψ be any two modules functions. Then the following are equivalent:

$$(i) \quad \lim_{x \rightarrow 0} \frac{\phi(x)}{\psi(x)} = \delta, \quad \lim_{x \rightarrow \infty} \frac{\phi(x)}{\psi(x)} = \epsilon, \text{ for some } \epsilon, \delta \in (0, \infty).$$

(ii) $H(\phi) = H(\psi)$, and the identity map I is metrically bounded.

PROOF. (i) $\rightarrow$ (ii). From the assumption in (i), one can choose a and b in $(0, \infty)$ such that

$$\frac{\phi(x)}{\psi(x)} \geq r \quad \text{on } [0, a]$$

$$\frac{\phi(x)}{\psi(x)} \geq s \quad \text{on } (b, \infty)$$

for some $r, s \in (0, \infty)$. Theorem 3.2 implies that $H(\phi) = H(\psi)$.

Let $f \in H(\phi)$. Consider the following sets:

$$E(a) = \{t: 0 \leq |f(e^{it})| < a\}$$

$$E(b) = \{t: |f(e^{it})| > b\}$$

$$E(a,b) = \{t: a \leq |f(e^{it})| \leq b\}.$$

Then:

$$\begin{aligned} \|f\|_{\psi} &= \int_{E(a)} \psi |f(e^{it})| dt + \int_{E(a,b)} \psi |f(e^{it})| dt + \int_{E(b)} \psi |f(e^{it})| dt \\ &\leq \frac{1}{r} \|f\|_{\phi} + \int_{E(a,b)} \psi |f(e^{it})| dt + \frac{1}{s} \|f\|_{\phi}. \end{aligned}$$

On the closed interval $[a,b]$, the continuity of $\frac{\phi(x)}{\psi(x)}$ implies the existence of $\lambda > 0$ such that $\psi(x) \leq \lambda \phi(x)$. Hence

$$\int_{E(a,b)} \psi |f(e^{it})| dt \leq \frac{1}{\lambda} \|f\|_{\phi}.$$

Thus, $\|f\|_{\psi} \leq \beta \|f\|_{\phi}$ where $\beta = \max(\frac{1}{r}, \frac{1}{s}, \frac{1}{\lambda})$. In a similar way one can show that $\|f\|_{\phi} \leq \gamma \|f\|_{\psi}$ for all $f \in H(\phi) = H(\psi)$. Hence the identity map is metrically bounded.

Conversely, (ii) $\rightarrow$ (i). Assume $H(\phi) = H(\psi)$ and $I: H(\phi) \leftrightarrow H(\psi)$ is metrically bounded. Then there exists α and β in $(0, \infty)$ such that

$$\|f\|_{\phi} \leq \alpha \|f\|_{\psi} \leq \|f\|_{\phi}.$$

Hence $\frac{\alpha}{\beta} \leq \frac{\|f\|_{\phi}}{\|f\|_{\psi}} \leq \alpha$ for all $f \in H(\phi) = H(\psi)$. Consider the function $f(z) = xz$ for $z = e^{it}$, $x \in (0, \infty)$. Then

$$\|f\|_{\phi} = \phi(x) \quad \text{and} \quad \|f\|_{\psi} = \psi(x).$$

Consequently $\frac{\alpha}{\beta} \leq \frac{\phi(x)}{\psi(x)} \leq \alpha$. Since $\alpha, \beta \in (0, \infty)$, (i) then follows. This ends the proof.

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