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STABILITY ANALYSIS OF LINEAR MULTISTEP METHODS FOR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. Stability properties of linear multistep methods for delay differential equations with respect to the test equation

$$y'(t) = ay(\lambda t) + by(t), \qquad t \geq 0,$$

$0 < \lambda < 1$, are investigated. It is known that the solution of this equation is bounded if and only if $|a| < -b$ and we examine whether this property is inherited by multistep methods with Lagrange interpolation and by parametrized Adams methods.

KEY WORDS AND PHRASES: Linear multistep method, delay differential equation, stability analysis.
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1. INTRODUCTION

Consider the delay differential equation (DDE)

$$\begin{aligned} y'(t) &= f(t,y(t),y(a(t))), \qquad t \leq t_0, \\ y(t) &= g(t), \qquad t_0 - \tau \leq t \leq t_0, \end{aligned} \tag{1.1}$$

$\tau \geq 0$, where $f:[t_0,\infty) \times R^2 \to R$ and $a:[t_0,\infty) \to R$ are continuous and $t_0 - \tau \leq a(t) \leq t$. Such equations arise in a number of practical applications such as, for example, control theory, electrodynamics, viscoelasticity, biomathematics, and medical sciences (compare Hale [1]).

The convergence theory of numerical methods for DDEs is rather well developed and resembles the convergence theory of corresponding numerical methods for ordinary differential equations (ODEs) (see Tavernini [2-4], Jackiewicz [5-7], Zverkina [8]). In this paper we wish to concentrate on stability analysis of some methods for (1.1).

To investigate the stability properties of numerical methods for (1.1), these methods are usually applied, with a fixed positive step size h, to various test equations with known region of stability. The simplest test equation is

$$\begin{aligned} y'(t) &= qy(t-\tau), \qquad t \leq 0, \\ y(t) &= g(t), \qquad -\tau \leq t \leq 0, \end{aligned} \tag{1.2}$$

$\tau \geq 0$, q - real, and it is known (see Bellman and Cooke [9]) that the solution of

this equation tends to zero as $t \to \infty$ for all q if and only if $q \in (-\pi/2\tau, 0)$. The numerical method, with step size $h = \tau/m$, where m is a positive integer, which inherits this property is said to be DA_0-stable and Cryer [10] gives some necessary and sufficient conditions for linear multistep method to be DA_0-stable. He also introduced a more general notion of GDA_0-stability which relaxes the condition $h = \tau/m$ to $h = \tau/(m-u)$, where $u \in [0,1)$, and investigated the properties of linear multistep methods with respect to this concept. Barwell [11] generalized some of Cryer's results to the case where q is complex and also considered a more general test equation

$$\begin{aligned} y'(t) &= py(t) + qy(t-\tau), & t &\geq 0, \\ y(t) &= g(t), & -\tau &\leq t \leq 0, \end{aligned} \tag{1.3}$$

with p and q complex. He introduced the notion of Q- and GQ-stability related to (1.2) (with q complex) and P- and GP- stability related to (1.3) which are analogues of Cryer's DA_0- and GDA_0-stability and investigated the stability properties of some simple multistep methods coupled with Lagrange interpolation. Still more general concepts of stability with respect to (1.2) were introduced by van der Houwen and Sommeijer [12] and stability criteria were derived for a class of linear multistep methods.

Stability analysis of numerical methods for DDEs is difficult since it is necessary to consider difference equations of arbitrarily high order. To illustrate this point suppose that the linear multistep method

$$\begin{aligned} &\sum_{j=0}^{k} \alpha_j y_h(t_{i+k-j}) = h \sum_{j=0}^{k} \beta_j f(t_{i+k-j}, y_h(t_{i+k-j}), y_h(a(t_{i+k-j}))), \\ &y_h(s) = g_h(s), \qquad s \in [t_0-\tau, t_0], \end{aligned} \tag{1.4}$$

$i=0,1,\ldots,$ coupled with Lagrange interpolation of sufficiently high order, and step size $h = \tau/m$, where m is a positive integer, is applied to (1.3). Here, $t_i = t_0 + ih$, $i=0,1,\ldots;$ y_h is an approximate solution; and g_h is an approximation to g. This leads to the difference equation

$$\begin{aligned} &\sum_{j=0}^{k} \alpha_j y_h(t_{i+k-j}) = h \sum_{j=0}^{k} \beta_j (py_h(t_{i+k-j}) + qy_h(t_{i+k-j-m})), \\ &y_h(s) = g_h(s), \qquad s \in [t_0-\tau, t_0], \end{aligned} \tag{1.5}$$

$i=0,1,\ldots,$ of order $m+k$. Now in order to establish some stability properties of (1.4) we should be able to decide whether or not the solution y_h of (1.5) is bounded (or tends to zero as $t \to \infty$). This is, in general, a nontrivial task and satisfactory results were obtained only for simplest numerical methods. Barwell [11] established P- and GP-stability for first and second order backward differentiation methods and Jackiewicz [13] determined stability regions for Θ-methods. Similar results with respect to the test equation (1.2) were obtained by Cryer [10] and Barwell [11]. Cryer proved that the Θ-methods are DA_0-stable if and only if $\Theta \in [\frac{1}{2},1]$ and that the Backward Euler method and the trapezoidal method are GDA_0-stable. Barwell proved that the Backward Euler method is Q-stable and conjectured that it is GQ-stable. For more general methods some stability results were established by Wiederholt [14], Al-Mutib [15], and Oppelstrup [16]. Wiederholt determined numerically (via boundary locus

method) the set of all (hp,hq) such that y_h (given by (1.5)) tends to zero as $t \to \infty$ for second order Milne predictor-corrector method and third order Adams predictor-corrector method for m=1,2, and 3. Similar results were obtained by Al-Mutib for the Runge-Kutta-Merson method, the trapezium rule and the fourth order implicit Runge-Kutta method. Oppelstrup investigated stability properties of Runge-Kutta-Fehlberg method combined with Hermite-Birkhoff interpolation with respect to the test equation (1.2).

More general test equations for functional differential equations were considered by Bickart [17] and Brayton and Willoughby [18].

It is the purpose of this paper to present stability analysis of some linear multistep methods based on the test equation

$$y'(t) = ay(\lambda t) + by(t), \qquad t \geq 0,$$
$$y(0) = y_0, \tag{1.6}$$

where a, b, and λ are real and $0 < \lambda < 1$. The importance of such equations in practical applications is discussed, for example, by Fox et al [19]. It follows from the results of Kato and McLeod [20] (see also Fox et al [19]) that all solutions of (1.6) are bounded if and only if $|a| < -b$ and we investigate whether this property is inherited by the approximate solution y_h when (1.4) is applied to (1.6). The problem is that the application of (1.4) to (1.6) leads to difference equations which are not of fixed order. A a consequence, the approach of Barwell [11], Cryer [10], Jackiewicz [13] is not applicable (compare the discussion of this topic in Jackiewicz [13]), and a different method of attack is needed. Roughly speaking, the approach of this paper consists of the following. Assuming that $y_h(t_j)$ for $j \leq i$ are bounded by a constant M and $\|y_h\|_{[t_0,t_i]}$ is bounded by γM for some $\gamma \geq 1$, we are looking for conditions such that $y_h(t_{i+1})$ and $\|y_h\|_{[t_i,t_{i+1}]}$ are bounded by the same constants M and γM respectively. Denoting the set of all (hb,ha) for which this is true by A_γ, we obtain a lower bound on the stability region with respect to (1.6) in the form $\bigcup_{\gamma \geq 1} A_\gamma$. This approach works reasonably well for simple multistep methods such as, for example Θ-methods. We were also able to obtain some sufficient stability conditions for linear multistep methods with Lagrange interpolation (Section 2) and for low order parametrized Adams-Bashforth and Adams-Moulton methods (Section 3). Unfortunately, this approach does not carry over to general linear multistep methods with Hermite interpolation and high order parametrized Adams methods.

As a byproduct of our analysis we established that the stability region for Θ-methods with respect to (1.3) is contained in the stability region with respect to (1.6). This partially answers the conjecture posed in Jackiewicz [13].

2. STABILITY ANALYSIS OF LAGRANGE INTERPOLATORY EXTENSIONS OF LINEAR MULTISTEP METHODS

A Lagrange interpolatory extension of the linear multistep method for ODEs with coefficients α_j, β_j, $j=0,1,\ldots,k$, is the method defined by

$$\sum_{j=0}^{k} \alpha_j y_h(t_{i+k-j}) = h \sum_{j=0}^{k} \beta_j F(t_{i+k-j}), \tag{2.1}$$

and

$$y_h(t_{i+k-1}+rh) = \sum_{j=0}^{k} U_j(r) y_h(t_{i+k-j}), \tag{2.2}$$

$i=0,1,\ldots,$ $r \in [0,1]$, where $F(t_\nu) = f(t_\nu, y_h(t_\nu), y_h(a(t_\nu)))$ and (2.2) is Lagrange interpolation formula, i.e. $U_0(1) = 1$, $U_j(1) = 0$, $j=1,2,\ldots,k$, $U_1(0) = 1$, $U_j(0) = 0$, $j \neq 1$; and $\sum_{j=0}^{k} U_j(r) = 1$ (compare Tavernini [3], where a more general Hermite interpolatory extension is defined). This method can be written in the form

$$a_0 y_h(t_{i+k-1}+rh) + \sum_{j=1}^{k} a_j(r) y_h(t_{i+k-j}) = h \sum_{j=0}^{k} b_j(r) F(t_{i+k-j}), \tag{2.3}$$

$i=0,1,\ldots,$ $r \in [0,1]$, where $a_0 = \alpha_0$; $\alpha_j(r) = \alpha_j U_0(r) - \alpha_0 U_j(r)$, $j=1,2,\ldots,k$; $b_j(r) = \beta_j U_0(r)$, $j=0,1,\ldots,k$. Denote by y_h^λ the approximate solution obtained when the method (2.3) is applied to (1.6). We propose the following definition.

DEFINITION. For given values of $x = hb$ and $y = ha$, the method (2.3) is said to be absolutely stable, if y_h^λ is bounded for $0 < \lambda < 1$. A region of absolute stability is the set of all points (x,y) such that the method (2.3) is absolutely stable. The method (2.3) is said to be A-stable if the region of absolute stability includes the stability region for (1.6), i.e. the set $\{(x,y): |y| < -x\}$.

We have the following:

THEOREM 1. Assume that (x,y) satisfies the inequality

$$\sum_{j=0}^{k} |-\alpha_j + x\beta_j| + \gamma|y| \sum_{j=0}^{k} |\beta_j| \leq |\alpha_0 - x\beta_0|, \tag{2.4}$$

where $\gamma = \sup\{\sum_{j=0}^{k} |U_j(r)|: r \in [0,1]\}$. Then the method (2.3) is absolutely stable.

PROOF. It is clear from (2.2) that if $|y_h^\lambda(t_j)|$ is bounded by a constant M for $j \leq i$ then $\|y_h^\lambda\|_{[0,t_i]}$ is bounded by γM. Here, $\|y_h^\lambda\|_{[0,t_i]} = \sup\{|y_h(s)|: s \in [0,t_i]\}$. Assume that $|y_h(t_j^\lambda)| \leq M$ for $j \leq i$. Then it is easy to check, using (2.1) with $F(t) = ay_h^\lambda(\lambda t) + by_h^\lambda(t)$, that $|y_h(t_{i+1})| \leq M$ and $\|y_h\|_{[0,t_{i+1}]} \leq \gamma M$ provided (2.4) holds, and the proof is complete.

To illustrate this theorem we apply it now to some simple methods for DDEs.

EXAMPLE 1. Consider the Θ-methods with linear interpolation for DDEs (1.1). These are the methods of the form

$$y_h(t_{i+1}) = y_h(t_i) + rh[\Theta F(t_{i+1}) + (1-\Theta)F(t_i)],$$
$$y_h(t_i+rh) = (1-r)y_h(t_i) + ry_h(t_{i+1}),$$

$i=0,1,\ldots,$ $r \in [0,1]$. Now $\gamma = 1$ and the inequality (2.4) takes the form

$$|1 + (1-\Theta)x| + |y| \leq |1 - \Theta x|.$$

It is easy to verify that the solution of this inequality is $\mathcal{R}_1$ for $0 \leq \Theta \leq \frac{1}{2}$ and $\mathcal{R}_1 \cup \mathcal{R}_2$ for $\frac{1}{2} < \Theta \leq 1$, where

$$\mathcal{R}_1 = \{(x,y): |y| \leq -x \text{ and } |y| \leq 2 + (1-2\Theta)x\}$$

and

$$\mathcal{R}_2 = \{(x,y): |y| \leq -2 + (2\Theta-1)x\}.$$

(In particular, the Backward Euler method ($\Theta = 1$) is A-stable). Comparing this with Jackiewicz [13], p.391, we see that the stability region of Θ-methods with respect to (1.3) is contained in the stability region of these methods with respect to (1.6), which partially answers the conjecture posed in Jackiewicz [13].

EXAMPLE 2. Consider the trapezoidal method with quadratic interpolation

$$y_h(t_i+rh) = \frac{1}{2}(r+1)(2-r)y_h(t_i) + \frac{r}{2}(r-1)y_h(t_{i+1}) + \frac{r}{4}(r+1)h[F(t_{i+1})+F(t_i)],$$

$i=0,1,\ldots,$ $r \in [0,1]$. This method can be written in the form

$$y_h(t_{i+1}) = y_h(t_i) + \frac{h}{2}[F(t_{i+1})+F(t_i)],$$

$$y_h(t_i+rh) = \frac{r}{2}(r+1)y_h(t_{i+1}) + (1-r^2)y_h(t_i) + \frac{r}{2}(r-1)y_h(t_{i-1}),$$

$i=0,1,\ldots,$ $r \in [0,1]$ (see Tavernini [3]). Now $\gamma = \frac{5}{4}$ and the inequality (2.4) reads

$$|1 + \frac{1}{2}x| + \frac{5}{4}|y| \le |1 - \frac{1}{2}x|.$$

The solution of this inequality is the set

$$R = \{(x,y): \ |y| \le -x \ \text{ and } \ |y| \le \frac{8}{5}\}.$$

For comparison, for the trapezoidal rule with linear interpolation (this corresponds to $\Theta = \frac{1}{2}$ in Example 1) the solution of (2.4) is

$$R^* = \{(x,y): \ |y| \le -x \ \text{ and } \ |y| \le 2\},$$

and $R \subset R^*$. This suggest that the trapezoidal rule with linear interpolation may have better stability properties than trapezoidal rule with quadratic interpolation.

EXAMPLE 3. Consider the method

$$y_h(t_{i-1}+rh) = y_h(t_{i-2}) + (1+r)hF(t_i),$$

$i=0,1,\ldots,$ $r \in [0,1]$ or, equivalently

$$y_h(t_i) = y_h(t_{i-2}) + 2hF(t_i),$$

$$y_h(t_{i-1}+rh) = \frac{1+r}{2}y_h(t_i) + \frac{1-r}{2}y_h(t_{i-2}),$$

$i=0,1,\ldots,$ $r \in [0,1]$. This method is not of type (2.3), but the same approach can be used to obtain a sufficient condition for absolute stability. It is clear that $\gamma = 1$ and inequality (2.4) takes the form

$$1 + 2|y| \le |1 - 2x|.$$

Therefore, this method is absolutely stable if $(x,y) \in A_1 \cup A_2$, where

$$A_1 = \{(x,y): \ |y| \le -x\},$$

$$A_2 = \{(x,y): \ |y| \le x - 1\}.$$

In particular, it is A-stable.

EXAMPLE 4. Consider the backward differentiation methods:

$$\sum_{j=0}^{k} \alpha_{k,j} y_h(t_{i+k-j}) = hF(t_{i+k}),$$

$$y_h(t_{i+k-1}+rh) = \sum_{j=0}^{k} U_{k,j}(r) y_h(t_{i+k-j}),$$

$i=0,1,\ldots,$ $r \in [0,1]$, $k=1,2,\ldots,6$, where

$$\alpha_{k,0} = \sum_{i=1}^{k} \frac{1}{i}, \qquad \alpha_{k,j} = \frac{(-1)^j}{j}\binom{k}{j}, \qquad j=1,2,\ldots,k,$$

$$U_{k,j}(r) = \prod_{\substack{m=0 \\ m\neq j}}^{k} \frac{r+m-1}{m-j}, \qquad k=1,2,\ldots, \quad j=1,2,\ldots,k$$

(the case k=1 corresponds to $\Theta=1$ in Example 1). Now inequality (2.4) reads

$$\sum_{j=1}^{k} |\alpha_{k,j}| + \gamma_k |y| \le |\alpha_{k,0} - x|,$$

$\gamma_k := \sup\{ \sum_{j=0}^{k} |\alpha_{k,j}| :\ r \quad [0,1]\}$, and its solution is

$$A_k = A_{k,1} \cup A_{k,2},$$

where

$$A_{k,1} = \{(x,y):\ x + \gamma_k |y| \le \sum_{j=1}^{k} \frac{1}{j}(1-\binom{k}{j})\},$$

$$A_{k,2} = \{(x,y):\ x - \gamma_k |y| \le \sum_{j=1}^{k} \frac{1}{j}(1+\binom{k}{j})\}$$

(note that $\sum_{j=1}^{k} \frac{1}{j}(1-\binom{k}{j}) < 0$ for $k \ge 2$). Unfortunately, this approach cannot be used to determine the stability of these methods in the neighborhood of the origin for $k \ge 2$.

3. STABILITY ANALYSIS OF PARAMETRIZED ADAMS METHODS

We will consider the (explicit) Adams-Bashforth and the (implicit) Adams-Moulton formulas. The Adams-Bashforth methods are given by

$$y_h(t_{i+k-1}+rh) = y_h(t_{i+k-1}) + h \sum_{j=0}^{k} b_{k,j}(r) F(t_{i+k-1-j}), \tag{3.1}$$

$i=0,1,\ldots,\quad r \in (0,1]$, where

$$b_{k,j}(r) = \int_0^r \Big(\prod_{\substack{m=0 \\ m\neq j}}^{k} \frac{s+m}{m-j} \Big) ds. \tag{3.2}$$

The method (3.1) can also be written in the form

$$y_h(t_{i+k-1}+rh) = y_h(t_{i+k-1}) + h \sum_{j=0}^{k} c_j(r) \nabla^j F(t_{i+k-1}), \tag{3.3}$$

$i=0,1,\ldots,\quad r \in (0,1]$, where ∇^j is the backward difference operator of order j. The coefficients c_j, $j=0,1,\ldots,k$, are independent of k, and are given by

$$c_j(r) = (-1)^j \int_0^r \binom{-s}{m} ds.$$

Using the generating function

$$G(t,r) := \sum_{m=0}^{\infty} c_m(r) t^m = \frac{1-(1-t)^{-r}}{\ln(1-t)}$$

we also have

$$c_0(r) = r, \qquad \sum_{j=0}^{m} \frac{1}{j+1} c_{m-j}(r) = (-1)^{m+1} \binom{-r}{m+1}. \tag{3.4}$$

Expressing $\nabla^j F(t_{i+k-1})$ in (3.3) in terms of $F(t_{i+k-1-\nu})$ and comparing (3.3) and (3.1) we can easily find the following relationship between $b_{k,j}$ and c_j:

$$b_{k,j}(r) = (-1)^j \sum_{m=j}^{k} \binom{m}{j} c_m(r). \tag{3.5}$$

Consider now the Adams-Moulton formulas for DDEs. These methods read

$$y_h(t_{i+k-1}+rh) = y_h(t_{i+k-1}) + h \sum_{j=0}^{k} b^*_{k,j}(r) F(t_{i+k-j}), \tag{3.6}$$

$i=0,1,\ldots,\quad r \in (0,1]$, where

$$b^*_{k,j}(r) = \int_0^r \Big(\prod_{\substack{m=0 \\ m\neq j}}^{k} \frac{s+m-1}{m-j} \Big) ds. \tag{3.7}$$

Another representation of (3.6) is

$$y_h(t_{i+k-1}+rh) = y_h(t_{i+k-1}) + h\sum_{j=0}^{k} c_j^*(r)\ {}^jF(t_{i+k}), \tag{3.8}$$

$i=0,1,\ldots,$ $r \in (0,1]$, where the coefficients c_j^*, $j=0,1,\ldots,k.$ are given by

$$c_j^*(r) = (-1)^j \int_{-1}^{-1+r} \binom{-s}{j} ds.$$

Using the generating function

$$G^*(t,r) = \sum_{m=0}^{\infty} c_m^*(r)t^m = \frac{(1-t)-(1-t)^{1-r}}{\ln(1-t)}$$

we also have

$$c_0^*(r) = r, \qquad \sum_{j=0}^{m} \frac{1}{j+1} c_{m-j}^*(r) = (-1)^{m-1}\binom{1-r}{m+1} \tag{3.9}$$

(compare Tavernini [3]). The relationship between $b_{k,j}^*$ and c_j^* is

$$b_{k,j}^*(r) = (-1)^j \sum_{m=j}^{k} \binom{m}{j} c_m^*(r). \tag{3.10}$$

Now we list some properties of the coefficients of the Adams-Bashforth and Adams-Moulton methods which will be useful later on. To be consistent with Henrici [21] we will use the following notation:

$$\beta_{k,j} := b_{k,j}(1), \qquad \gamma_j := c_j(1), \qquad \beta_{k,j}^* := b_{k,j}^*(1), \qquad \gamma_j^* := c_j^*(1).$$

P1. $\gamma_m > 0, \quad m \le 0.$

P2. $\gamma_0^* = 1, \quad \gamma_m^* < 0, \quad m \ge 1.$

P3. $\sum_{i=0}^{m} \gamma_i^* = \gamma_m, \quad m \ge 0.$

P4. $\beta_{k,0} > 0, \quad k \ge 0.$

P5. $u_k := \beta_{k,0} - \sum_{j=1}^{k} |\beta_{k,j}| > 0, \quad k=0,1,2; \quad u_k < 0, \quad k \ge 3.$

P6. $\beta_{k,0}^* > 0, \quad \beta_{k,1}^* > 0, \quad k \ge 1.$

P7. $\beta_{k,1}^* - \beta_{k,0}^* > 0, \quad k \ge 1.$

P8. $w_k^* := \beta_{k,0}^* - \sum_{j=1}^{k} |\beta_{k,j}| = 0, \quad k = 1; \quad w_k^* < 0, \quad k \ge 2.$

P9. $u_k^* := \beta_{k,0}^* + \beta_{k,1}^* - \sum_{j=2}^{k} |\beta_{k,j}^*| > 0, \quad k=1,2,3,4,5; \quad u_k^* < 0, \quad k \ge 6.$

P10. $\|b_{k,j}\|_{[0,1]} = |b_{k,j}(1)|; \quad \|b_{k,j}^*\|_{[0,1]} = |b_{k,j}(1)|, \quad k \ge 0, \quad j=0,1,\ldots,k.$

The properties P1-P3 are well known (compare Henrici [21]), where the first few values of γ_m and γ_m^* are also listed, or Hall [22]). P4 follows from (3.5) for $r=1$, $j=0$, and P1. It is easy to check, using (3.5) for $r=1$, that

$$u_k = \sum_{m=0}^{k} \gamma_m - \sum_{m=0}^{k} \sum_{m=j}^{k} \binom{m}{j}\gamma_m = 1 + \sum_{m=1}^{k} \gamma_m - \sum_{m=1}^{k} \sum_{j=1}^{k} \binom{m}{j}\gamma_m = 1 + \sum_{m=1}^{k} (2-2^m)\gamma_m.$$

Hence, $u_0 = 1$, $u_1 = 1$, $u_2 = \frac{1}{6}$, $u_3 = -\frac{25}{12}$, and $u_{k+1} < u_k$ for $k \ge 3$ which establishes P5. P6 follows from (3.10) for $r=1$, $j=0,1$, and P1-P3. Using (3.10) for $r=1$ we obtain

$$\beta_{k,1}^* - \beta_{k,0}^* = -\sum_{i=2}^{k} (1+i)\gamma_i^*$$

which, together with P2, establishes P7. In view of (3.10) for r=1 we have

$$w_k^* = \sum_{m=0}^{k} \gamma_m^* + \sum_{m=1}^{k} m\gamma_m^* + \sum_{j=2}^{k}\sum_{m=j}^{k} \binom{m}{j}\gamma_m^* = \gamma_0^* + 2\gamma_1^* + \sum_{m=2}^{k}(1+m)\gamma_m^*$$

$$+ \sum_{m=2}^{k}\sum_{j=2}^{k}\binom{m}{j}\gamma_m^* = \sum_{m=2}^{k}(1+m+2^m-m-1)\gamma_m^* = \sum_{m=2}^{k} 2^m\gamma_m^*$$

which proves P8. Similarly,

$$u_k^* = 1 + \sum_{m=2}^{k}(2^m-2m)\gamma_m^*.$$

It is clear that $u_k^* > 0$ for $k=1,2,3,4,5$, $u_6^* < 0$ and $u_{k+1}^* < u_k^*$ for $k \geq 6$ which proves P9. P10 follows easily from (3.2) and (3.7).

We are now in a position to formulate our stability theorems.

THEOREM 2. Denote by A_γ the set of all $(x,y) = (hb,ha)$ such that

$$|1 + \beta_{k,0}x| + \sum_{j=1}^{k}|\beta_{k,j}||x| + \gamma\sum_{j=0}^{k}|\beta_{k,j}||y| \leq 1 \tag{3.11}$$

and

$$\sum_{j=0}^{k}|\beta_{k,j}|(|x| + \gamma|y|) \leq \gamma - 1, \tag{3.12}$$

and put $A = \bigcup_{\gamma\geq 1} A_\gamma$. Then if $(x,y) \in A$ the Adams-Bashforth method (3.1) is absolutely stable.

PROOF. The method (3.1) applied to (1.6) takes the form

$$y_h^\lambda(t_{i+k-1}+rh) = y_h^\lambda(t_{i+k-1}) + h\sum_{j=0}^{k} b_{k,j}(r)(ay_h^\lambda(\lambda t_{i+k-1-j}) + by_h^\lambda(t_{i+k-1-j})), \tag{3.13}$$

$i=0,1,\ldots,$ $r \in (0,1]$. Let M be a constant such that $\|y_h^\lambda\|_{[0,t_{k-1}]} \leq M$. Assume that $|y_h^\lambda(t_j)| \leq M$ for $j=0,1,\ldots,i+k-1$, and $\|y_h^\lambda\|_{[0,t_{i+k-1}]} \leq \gamma M$, where $\gamma \geq 1$. Then, using (3.13) for r=1 it follows that $|y_h^\lambda(t_{i+k})| \leq M$ if (3.11) is satisfied. Similarly, using (3.13) and P10 we can see that $\|y_h^\lambda\|_{[0,t_{i+k}]} \leq \gamma M$ if (3.12) is satisfied. Therefore, if $(x,y) \in A$, the approximate solution y_h^λ given by (3.13) is bounded and the theorem follows.

Define u_k as in P5 and put $v_k := \sum_{j=0}^{k}|\beta_{k,j}|$. It is easy to see, using P4, that A_γ can be written in the form

$$A_\gamma = B_\gamma \cap C_\gamma,$$

where

$$B_\gamma = \{(x,y):\ x \leq 0,\quad u_k x + \gamma v_k|y| \leq 0,\quad v_k(-x + \gamma|y|) \leq 2\}$$

and

$$C_\gamma = \{(x,y):\ x \leq 0,\quad v_k(-x + \gamma|y|) \leq \gamma - 1\}.$$

It is clear in view of P5 that B_γ is empty for $k \geq 3$, therefore this theorem applies only for k=0,1, and 2. It follows after some computations that the boundary of A for k=0,1, and 2 is given by

$$y = \begin{cases} \pm(2+v_k x)/(3v_k), & -2/v_k \leq x \leq -1/\beta_{k,0}, \\ \pm u_k x/(v_k(2\beta_{k,0}x-1)), & -1/\beta_{k,0} \leq x \leq 0. \end{cases}$$

These boundaries for $y \geq 0$ are plotted in Fig. 1.

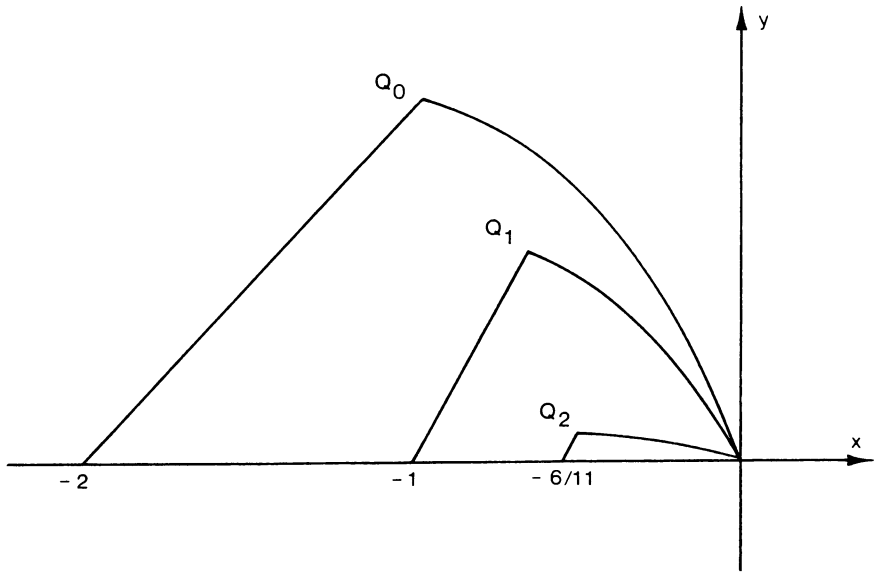

Fig. 1. Stability regions for Adams-Bashforth methods for k=0,1,2.

The coordinates of the points $Q_i(x_i,y_i)$, $i=0,1,2$, are given in Table 1 below.

k	x_k	y_k
0	-1	1/3
1	-2/3	1/4
2	-12/23	2/253

Table 1

For k=0 we already obtained a better bound on stability region in Example 1 for $\Theta=0$.

Our next theorem deals with the Adams-Moulton methods (3.6).

THEOREM 3. Denote by A^*_γ the set of all $(x,y) = (hb,ha)$ such that

$$|1 + \beta^*_{k,1}x| + \sum_{j=2}^{k} |\beta^*_{k,j}||x| + \gamma \sum_{j=0}^{k} |\beta^*_{k,j}||y| \leq |1 - \beta^*_{k,0}x| \tag{3.14}$$

and

$$\sum_{j=0}^{k} |\beta^*_{k,j}|(|x| + \gamma|y|) \leq \gamma - 1, \tag{3.15}$$

and put $A^* = \bigcup_{\gamma\geq 1} A_\gamma$. Then if $(x,y) \in A^*$ the Adams-Moulton method (3.6) is absolutely stable.

PROOF. The proof is similar to that of Theorem 2. The method (3.6) applied to (1.6) takes the form

$$y^\lambda_h(t_{i+k-1}+rh) = y^\lambda_h(t_{i+k-1}) + h \sum_{j=0}^{k} b^*_{k,j}(r)(ay^\lambda_h(\lambda t_{i+k-j}) + by^\lambda_h(t_{i+k-j})), \tag{3.16}$$

$i=0,1,\ldots,$ $r \in (0,1]$. For $r=1$ this can be written in the form

$$(1-hb\beta^*_{k,0})y^\lambda_h(t_{i+k}) = (1+hb\beta^*_{k,1})y^\lambda_h(t_{i+k-1}) + ha \sum_{j=0}^{k} \beta^*_{k,j}y^\lambda_h(\lambda t_{i+k-j}) + hb \sum_{j=2}^{k} \beta^*_{k,j}y^\lambda_h(t_{i+k-j}). \tag{3.17}$$

Denote by i_0 the smallest integer such that $t_{i_0+k} \le t_{i_0+k}$ and let M be a constant such that $\|y_h^\lambda\|_{[0,t_{i_0+k}]} \le M$. Assume that $|y_h^\lambda(t_j)| \le M$ for $j=0,1,\ldots,$ $i+k-1$, $i \ge i_0$, and $\|y_h^\lambda\|_{[0,t_{i+k-1}]} \le \gamma M$, where $\gamma \ge 1$. Then it is easy to check using (3.17) that $|y_h(t_{i+k})| \le M$ if (3.14) is satisfied. Similarly, using (3.16) and P10 it follows that $\|y_h\|_{[0,t_{i+k}]} \le \gamma M$ if (3.15) holds. Therefore, if $(x,y) \in A^*$, the approximate solution y_h^λ defined by (3.16) is bounded, which is our claim.

For k=0 (this corresponds to the backward Euler method) condition (3.14) and (3.15) take the form

$$1 + \gamma|y| \le |1 - x|,$$
$$|x| + \gamma|y| \le \gamma - 1,$$

and it can be verified that the boundary of the region A^* is given by

$$y = \begin{cases} \pm x/(2x-1), & x \le 0, \\ \pm(x-2)/(2x-1), & x \ge 2. \end{cases}$$

We obtained a better bound on stability region in Example 1 for $\Theta=1$. To get some idea how the region A_γ^* looks like for $k \ge 1$ we rewrite (3.14) and (3.15) in a different form. Define w_k^* and u_k^* as in P8 and P9 and put $v_k^* := \sum_{j=0}^{k} |\beta_{k,j}^*|$. It can be checked by considering special cases and using P6-P8 that

$$A_\gamma^* = B_\gamma^* \quad C_\gamma^*,$$

where

$$B_\gamma^* = \{(x,y):\ x \le 0,\ u_k^* x + \gamma v_k^* |y| \le 0,\ w_k^* x + \gamma v_k^* |y| \le 2\}$$

and

$$C_\gamma^* = \{(x,y):\ x \le 0,\ v_k^*(-x + \gamma|y|) \le \gamma - 1\}.$$

In view of P9 the set B_γ^* is empty for $k \ge 6$, therefore, Theorem 3 is applicable only for $k \le 5$. It follows after some computations that the boundary of the region A^* for $k=1,2,3,4,$ and 5 is given by

$$y = \begin{cases} \pm(2-w_k^*)/((v_k^*(3-(w_k^*+v_k^*)x)), & 2/w_k^* \le x \le -1/\beta_{k,1}^*, \\ \pm u_k^* x/(v_k^*((u_k^*+v_k^*)x-1)), & -1/\beta_{k,1}^* \le x \le 0. \end{cases}$$

Using the table of coefficients of Adams-Moulton methods given in Lapidus and Seifeld [23] we can compute u_k^*, v_k^*, and w_k^* and these boundaries, for $k=1,2,3,4,5$, are plotted in Fig. 2. The coordinates of the points $Q_k^*(x_k^*,y_k^*)$, $k=1,2,3,4,5$, are given in Table 2 below

k	x_k^*	y_k^*
1	-2	2/5
2	-3/2	36/119
3	-24/19	88/425
4	-360/323	4562/39232
5	-1440/1427	303840/8845621

Table 2

As mentioned above, we were not able to apply the approach used in this paper to the general linear multistep methods with Hermite interpolation or high order Adams methods.

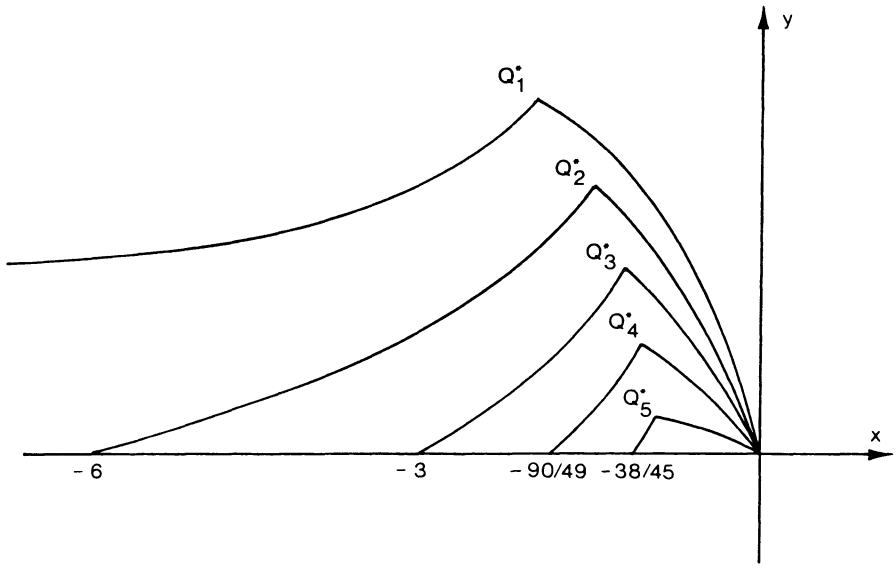

Fig. 2. Stability regions for Adams-Moulton methods for k=1,2,3,4,5.

REMARK. It was pointed out to us by Professor M. Zennaro that the main theorem in the paper by Jackiewicz [13] gives only sufficient, not necessary and sufficient condition for absolute stability of Θ-methods for delay differential equations.

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