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OSCILLATION THEOREMS FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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ABSTRACT. New oscillation criteria for the oscillatory behaviour of the differential

$$(a(t)\dot{x}(t))^{\cdot} + p(t)\dot{x}(t) + q(t)f(x[g(t)]) = 0\ , \qquad (^{\cdot} = \tfrac{d}{dt})$$

and

$$(a(t)\psi(x(t))\dot{x}(t))^{\cdot} + p(t)\dot{x}(t) + q(t)f(x[g(t)]) = 0\ ,$$

are established.

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1. INTRODUCTION.

This paper is concerned with the oscillatory behavior of solutions of second order nonlinear damped differential equations with deviating argument of the form

$$(a(t)\dot{x}(t))^{\cdot} + p(t)\dot{x}(t) + q(t)f(x[q(t)]) = 0\ , \tag{1.1}$$

and

$$(a(t)\,\psi(x(t))\dot{x}(t))^{\cdot} + p(t)\dot{x}(t) + q(t)f(x[g(t)]) = 0\ , \tag{1.2}$$

where a, g, p, q: $[t_0, \infty) \rightarrow [0, \infty)$, ψ, f: $\mathbb{R} \rightarrow \mathbb{R} = (-\infty, \infty)$ are continuous, $a(t) > 0$, $q(t)$ not identically zero on any ray of the form $[t^*, \infty)$ for some $t^* \geq t_0$ and $\lim_{t\to\infty} g(t) = \infty$.

We restrict our attention to those solutions of equations (1.1) and (1.2) which exist on some ray $[t_1, \infty)$, $t_1 \geq t_0$ and which are nontrivial in any neighborhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. An equation is called oscillatory if all its solutions are oscillatory.

In the study of the second order sublinear differential equation

$$\ddot{x}(t) + q(t)|x(t)|^{\alpha}\operatorname{sgn}x(t) = 0, \qquad 0 < \alpha < 1, \tag{1.3}$$

where $q: [t_0, \infty) \to R$ is continuous, there are many criteria for oscillation which involve the behavior of the integral of q. In particular, Belohorec [1] has shown that the condition

$$\int^{\infty} s^{\beta} q(s)\,ds = \infty \qquad \text{for some } \beta \in [0, \alpha]$$

is sufficient for the oscillation of equation (1.3), and Kamenev [6] has established that equation (3) is oscillatory if

$$\limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} (t - s) q(s)\,ds = \infty .$$

Recently, Kura [7] has presented a new criterion for the oscillation of equation (1.3) which improves upon those of Belohorec and Kamenev. Kura proved that a sufficient condition for the oscillation of equation (3) is that

$$\limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} (t - s) s^{\beta} q(s)\,ds = \infty \qquad \text{for some } \beta \in [0, \alpha] .$$

These results have been further extended by Philos [8] to a more general equation

$$\ddot{x}(t) + p(t)\dot{x}(t) + q(t) f(x(t)) = 0 ,$$

where $p, q: [t_0, \infty) \to R$, $f: R \to R$ are continuous, $xf(x) > 0$, $f'(x) > 0$ for $x \neq 0$ and f is strongly sublinear i.e. $\int_{\pm 0} \frac{du}{f(u)} < \infty$. The above results can be applied only to ordinary differential equations.

The purpose of this paper is to establish some new oscillation criteria for the differential equations (1.1) and (1.2). In fact, we impose no conditions on the function f other that $xf(x) > 0$ for $x \neq 0$ and nondecreasing. Thus our results can be applied to superlinear, linear and sublinear differential equations. We also like to mention that we do not stipulate that the function g in equations (1.1) and (1.2) is either retarded or advanced. Hence our theorems hold for ordinary, retarded, advanced and equations of mixed-type.

2. THE EQUATION (1.1).

We assume that

$$xf(x) > 0 \quad \text{and} \quad f'(x) \geq 0 \quad \text{for} \quad x \neq 0 \,, \qquad (' = \tfrac{d}{dx}) \,, \tag{2.1}$$

$$\int_T^\infty \frac{1}{a(s)} \exp\left(\int_T^s - \frac{p(u)}{a(u)} du \right) ds = \infty, \qquad T \geq t_0 \,. \tag{2.2}$$

Suppose further that there is a differentiable function

$$\sigma : [t_0, \infty) \to (0, \infty)$$

such that

$$\sigma(t) \leq g(t) \,, \quad \dot{\sigma}(t) \geq 0 \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad \lim_{t \to \infty} \sigma(t) = \infty \,. \tag{2.3}$$

THEOREM 1. Let conditions (2.1) - (2.3) hold and assume that there exists a twice differentiable function

$$\rho : [t_0, \infty) \to (0, \infty)$$

such that

$$\ddot{\rho}(t) \leq 0 \quad \text{and} \quad \frac{\dot{\rho}(t)}{\rho(t)} + \frac{a(t)}{\rho(t)} \left(\frac{\rho(t)}{a(t)} \right)^{\cdot} \leq \frac{p(t)}{a(t)} \quad \text{for} \quad t \geq t_0 \,. \tag{2.4}$$

If

$$\limsup_{t \to \infty} \frac{1}{t^n} \int_{t_0}^t (t-s)^n \frac{\rho(s)}{a(s)} q(s) ds = \infty \quad \text{for some} \quad n \geq 1 \,, \tag{2.5}$$

then equation (1.1) is oscillatory.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$ for $t \geq t_1$. By a Lemma in [5] and condition (2.2), there exists a $t_2 \geq t_1$ such that

$$\dot{x}(t) > 0 \quad \text{and} \quad \dot{x}[g(t)] > 0 \quad \text{for all } t \geq t_2 \,. \tag{2.6}$$

Now, define

$$\omega(t) = \rho(t) \int_{t_2}^t \frac{\dot{x}(s)}{f(x[\sigma(s)])} ds \,, \quad t \geq t_2 \,.$$

Then it is easy to verify that

$$\ddot{\omega}(t) = \ddot{\rho}(t) \int_{t_2}^t \frac{\dot{x}(s)}{f(x[\sigma(s)])} ds + \frac{\dot{\rho}(t)\dot{x}(t)}{f(x[\sigma(t)])} + \left(\frac{\rho(t)}{a(t)} \right)^{\cdot} \frac{a(t)\dot{x}(t)}{f(x[\sigma(t)])}$$
$$+ \frac{\rho(t)}{a(t)} \frac{(a(t)\dot{x}(t))^{\cdot}}{f(x[\sigma(t)])} - \rho(t)\dot{\sigma}(t) \frac{\dot{x}(t)\dot{x}[\sigma(t)] f'(x[\sigma(t)])}{f^2(x[\sigma(t)])} \tag{2.7}$$

$$= -\frac{\rho(t)}{a(t)}\, q(t)\, \frac{f(x[g(t)])}{f(x[\sigma(t)])} + \ddot{\rho}(t) \int_{t_2}^{t} \frac{\dot{x}(s)}{f(x[\sigma(s)])}\, ds$$

$$+ \left[\dot{\rho}(t) + a(t)\left(\frac{\rho(t)}{a(t)}\right)^{\cdot} - \frac{\rho(t)\,\sigma(t)}{a(t)}\right] \frac{\dot{x}(t)}{f(x[\sigma(t)])}$$

$$- \rho(t)\,\dot{\sigma}(t)\, \frac{\dot{x}(t)\,\dot{x}[\sigma(t)]\,f'(x[\sigma(t)])}{f^2(x[\sigma(t)])} \quad .$$

Using conditions (2.1), (2.3) and (2.6) we obtain

$$f(x[\sigma(t)]) \le f(x[g(t)]) \qquad \text{for} \quad t \ge t_2 ,$$

and by conditions (2.1), (2.3), (2.4) and (2.6) we have

$$\ddot{\omega}(t) \le -\frac{\rho(t)}{a(t)}\, q(t) \qquad \text{for every } t \ge t_2 \,. \tag{2.8}$$

Thus, for $t \ge t_2$ we have

$$\int_{t_2}^{t} (t-s)^n \frac{\rho(s)}{a(s)}\, q(s)\, ds \le -\int_{t_2}^{t} (t-s)^n\, \ddot{\omega}(s)\, ds$$

$$= (t-t_2)^n\, \dot{\omega}(t_2) + n(t-t_2)^{n-1}\, \omega(t_2)$$

$$- n(n-1) \int_{t_2}^{t} (t-s)^{n-2}\, \omega(s)\, ds$$

$$\le (t-t_2)^n\, \dot{\omega}(t_2) + n(t-t_2)^{n-1}\, \omega(t_2) \ .$$

Hence,

$$\frac{1}{t^n} \int_{t_2}^{t} (t-s)^n \frac{\rho(s)}{a(s)}\, q(s)\, ds \le \left(1-\frac{t_2}{t}\right)^n \dot{\omega}(t_2) + \frac{n\,\omega(t_2)}{t} \left(1-\frac{t_2}{t}\right)^{n-1} , \quad t \ge t_2 \,.$$

On the other hand, for $t \ge t_2$ we have

$$\frac{1}{t^n} \int_{t_0}^{t} (t-s)^n \frac{\rho(s)}{a(s)}\, q(s)\, ds - \frac{1}{t^n} \int_{t_2}^{t} (t-s)^n \frac{\rho(s)}{a(s)}\, q(s)\, ds$$

$$= \frac{1}{t^n} \int_{t_0}^{t_2} (t-s)^n \frac{\rho(s)}{a(s)}\, q(s)\, ds$$

$$\le \left(1 - \frac{t_0}{t}\right)^n \int_{t_0}^{t_2} \frac{\rho(s)}{a(s)}\, q(s)\, ds \ ,$$

and consequently

$$\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\frac{\rho(s)}{a(s)}q(s)ds \le \left(1-\frac{t_2}{t}\right)^n \omega(t_2) + \frac{n\omega(t_2)}{t}\left(1-\frac{t_2}{t}\right)^{n-1}$$

$$+\left(1-\frac{t_0}{t}\right)^n \int_{t_0}^{t_2}\frac{\rho(s)}{a(s)}q(s)ds, \qquad \text{for } t \ge t_2 .$$

This gives,

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\frac{\rho(s)}{a(s)}q(s)ds \le \omega(t_2) + \int_{t_0}^{t_2}\frac{\rho(s)}{a(s)}q(s)ds ,$$

which contradicts (2.5). This completes the proof of the theorem.

For illustration we consider the following examples.

EXAMPLE 1. The differential equation

$$\ddot{x}(t) + \left(\frac{1}{t}-1\right)\dot{x}(t) + \frac{1}{t(\ln[t+\sin t])^\alpha}|x[t+\sin t]|^\alpha \operatorname{sgn} x[t+\sin t] = 0 , \tag{2.9}$$

$$0 < \alpha \le 1 \quad \text{and} \quad t > e ,$$

has a nonoscillatory solution $x(t) = \ln t$. Only the damping coefficient $P(t)$ is negative for $t > e$, violating the assumptions of Theorem 1.

EXAMPLE 2. Consider the differential equation

$$\left(\frac{1}{t}\dot{x}(t)\right)^{\cdot} + \frac{2}{t^2}\dot{x}(t) + \frac{1}{t\sqrt{t}}|x[t+\cos t]|^\alpha \operatorname{sgn} x[t+\cos t] = 0 , \tag{2.10}$$

$$\alpha > 0 \quad \text{and} \quad t \ge t_0 \ge 2 .$$

We let $\rho(t) = \sqrt{t}$ and $\sigma(t) = t - 1$. The conditions of Theorem 1 are satisfied and so, all solutions of equation (2.10) are oscillatory.

EXAMPLE 3. The differential equation

$$\ddot{x}(t) + \frac{1}{t}\dot{x}(t) - t^{3\alpha-3}|x[t^3]|^\alpha \operatorname{sgn} x[t^3] = 0 , \ \alpha \ge \frac{1}{2} \text{ and } t \ge t_0 = 1 , \tag{2.11}$$

has a nonoscillatory solution $x(t) = \frac{1}{t}$. All the conditions of Theorem 1 are satisfied for $\rho(t) = t$ and $\sigma(t) = t^3$ except condition (2.5), since the function $q(t)$ is negative for $t \ge t_0$. On the other hand, the differential equation

$$\ddot{x}(t) + \frac{1}{t}\dot{x}(t) + t^{3\alpha-3}|x[t^3]|^\alpha \operatorname{sgn} x[t^3] = 0 , \ \alpha \ge \frac{1}{2} \text{ and } t \ge t_0 = 1 , \tag{2.12}$$

is oscillatory by Theorem 1 for $\rho(t) = t^{1/2}$ and $\sigma(t) = t^3$.

One can check that none of the oscillation criteria of [1] - [9] can describe the oscillatory character of either equation (2.10) for $\alpha > 0$ or equation (2.12) for $\alpha \ge \frac{1}{2}$.

The following theorem is concerned with the case when condition (2.4) fails.

We assume that

$$\int^{+\infty} \frac{du}{f(u)} < \infty \quad \text{and} \quad \int^{-\infty} \frac{du}{f(u)} < \infty . \tag{2.13}$$

It will be convenient to make use of the following notation: for any $t \geq t_0$ we let

$$\gamma(t, c) = \dot{\rho}(t) + a(t)\left(\frac{\rho(t)}{a(t)}\right)^{\cdot} - \frac{p(t)\rho(t)}{ca(t)} ,$$

where $\rho \in C^2[[t_0, \infty) , (0, \infty)]$.

THEOREM 2. Let conditions (2.1) - (2.3) and (2.13) hold and let $\sigma(t) \geq t$ for $t \geq t_0$ and the function ρ be as in Theorem 1 such that

$$\ddot{\rho}(t) \geq 0 , \quad \gamma(t, 1) \geq 0 , \quad \dot{\gamma}(t, 1) \leq 0 \quad \text{for} \quad t \geq t_0 \tag{2.14}$$

and

$$\limsup_{t\to\infty} \frac{t}{\rho(t)} < \infty.$$

If

$$\limsup_{t\to\infty} \frac{1}{\rho(t)} \int_{t_0}^{t} \int_{t_0}^{s} \frac{\rho(u)}{a(u)} q(u)\,du\,ds = \infty , \tag{2.15}$$

then equation (1.1) is oscillatory.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$ for $t \geq t_0$. As in the proof of Theorem 1 we get (2.7). Now, using (2.6) and the fact that $\sigma(t) \geq t$ for $t \geq t_2$ we obtain

$$\ddot{\omega}(t) \leq -\frac{\rho(t)}{a(t)} q(t) + \ddot{\rho}(t) \int_{t_2}^{t} \frac{\dot{x}(s)}{f(x(s))} ds + \gamma(t, 1) \frac{\dot{x}(t)}{f(x(t))} .$$

By condition (2.13), we have

$$\ddot{\omega}(t) \leq -\frac{\rho(t)}{a(t)} q(t) + C\ddot{\rho}(t) + \gamma(t, 1) \frac{\dot{x}(t)}{f(x(t))} ,$$

where $C = \int_{x(t_2)}^{\infty} \frac{du}{f(u)}$. Integrating the above inequality from t_2 to t we get

$$\dot{\omega}(t) \leq \dot{\omega}(t_2) - \int_{t_2}^{t} \frac{\rho(s)}{a(s)} q(s)\,ds + C\dot{\rho}(t) - C\dot{\rho}(t_2) + \int_{t_2}^{t} \gamma(s, 1) \frac{\dot{x}(s)}{f(x(s))} ds .$$

By the Bonnet theorem, for any $t \geq t_2$, there exists a $\xi \in [t_2, t]$ such that

$$\int_{t_2}^{t} \gamma(s, 1) \frac{\dot{x}(s)}{f(x(s))} ds = \gamma(t_2, 1) \int_{t_2}^{\xi} \frac{\dot{x}(s)}{f(x(s))} ds$$

$$= \gamma(t_2, 1) \int_{x(t_2)}^{x(\xi)} \frac{du}{f(u)}$$

$$\leq \gamma(t_2, 1) \int_{x(t_2)}^{\infty} \frac{du}{f(u)} .$$

Thus, for every $t \geq t_2$

$$\dot{\omega}(t) \leq K - \int_{t_2}^{t} \frac{\rho(s)}{a(s)} q(s)ds + C\dot{\rho}(t) , \tag{2.16}$$

where $K = \dot{\omega}(t_2) - C\dot{\rho}(t_2) + \gamma(t_2, 1) \int_{x(t_2)}^{\infty} \frac{du}{f(u)}$.

Integrating (2.16) from t_2 to t we have

$$\int_{t_2}^{t} \int_{t_2}^{s} \frac{\rho(s)}{a(s)} q(u)duds \leq C\rho(t) + Kt - \omega(t) + (\omega(t_2) - C\rho(t_2)) .$$

Dividing by $\rho(t)$ and taking limit superior of both side as $t \to \infty$, we obtain a contradiction to (2.15). This completes the proof of the theorem.

REMARK. It is easy to check that Theorem 2 is not applicable to equation (2.12) if $\frac{1}{2} \leq \alpha \leq 1$. On the other hand, Theorem 2 can be applied in some cases in which Theorem 1 is not applicable. Such a case is described in Example 4 below.

EXAMPLE 4. Consider the differential equation

$$\ddot{x}(t) + \frac{1}{t}\dot{x}(t) + \frac{c}{t^2}|x[g(t)]|^{\alpha} \operatorname{sgn} x[g(t)] = 0 , \quad t \geq t_0 > 0 , \tag{2.17}$$

where $c > 0$, $\alpha > 1$ and $\sigma(t) = g(t) \geq t$ with $\sigma(t) \geq 0$ for $t \geq t_0$.
The conditions of Theorem 2 are satisfied for $\rho(t) = t$ and hence equation (2.17) is oscillatory.

3. THE EQUATION.

In order to obtain results for equation (1.2) similar to those in section 2 we assume

$$0 < c \leq \psi(x) \leq c_1 \quad \text{for all } x , \tag{3.1}$$

$$\int_T^{\infty} \frac{1}{a(s)} \exp\left(\int_T^s - \frac{p(u)}{ca(u)} du\right) ds = \infty , \quad \text{for all } T \geq t_0 . \tag{3.2}$$

THEOREM 3. Let conditions (2.1), (2.3), (3.1) and (3.2) hold and let ρ be as in Theorem 1 such that

$$\ddot{\rho}(t) \leq 0 \quad \text{and} \quad \frac{\dot{\rho}(t)}{\rho(t)} + \frac{a(t)}{\rho(t)}\left(\frac{\rho(t)}{a(t)}\right)^{\dot{}} \leq \frac{1}{c_1} \frac{p(t)}{a(t)} \quad \text{for all } t \geq t_0 \tag{3.3}$$

If condition (2.5) holds, then equation (1.2) is oscillatory.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1.2). Assume that $x(t) > 0$ for $t \geq t_0$. There exists a $t_1 \geq t_0$ so that $x[\sigma(t)] > 0$ for $t \geq t_1$. The hypotheses of Lemma in [5] are satisfied and hence there exists a $t_2 \geq t_1$ such that

$$\dot{x}(t) > 0 \quad \text{and} \quad \dot{x}[\sigma(t)] > 0 \quad \text{for all } t \geq t_2 .$$

Now, we define

$$\omega(t) = \rho(t) \int_{t_2}^{t} \frac{\psi(x(s))\dot{x}(s)}{f(x[\sigma(s)])} ds , \quad \text{for} \quad t \geq t_2 .$$

Then for every $t \geq t_2$ we obtain

$$\ddot{\omega}(t) = -\frac{\rho(t)}{a(t)} q(t) \frac{f(x[g(t)])}{f(x[\sigma(t)])} + \ddot{\rho}(t) \int_{t_2}^{t} \frac{\psi(x(s))\dot{x}(s)}{f(x[\sigma(s)])} ds$$

$$+ \left[\dot{\rho}(t) + a(t)\left(\frac{\rho(t)}{a(t)}\right)^{\cdot} - \frac{p(t)\rho(t)}{a(t)} \frac{1}{\psi(x(t))}\right] \frac{\psi(x(t))\dot{x}(t)}{f(x[\sigma(t)])}$$

$$- \rho(t)\dot{\sigma}(t) \frac{\psi(x(t))f'(x[\sigma(t)])\dot{x}(t)\dot{x}[\sigma(t)]}{f^2(x[\sigma(t)])} .$$

It is easy to verify that

$$\ddot{\omega}(t) \leq -\frac{\rho(t)}{a(t)} q(t) + \left[\dot{\rho}(t) + a(t)\left(\frac{\rho(t)}{a(t)}\right)^{\cdot} - \frac{p(t)\rho(t)}{c_1 a(t)}\right] \frac{\psi(x(t))\dot{x}(t)}{f(x[\sigma(t)])}$$

$$\leq -\frac{\rho(t)}{a(t)} q(t) , \quad t \geq t_2 .$$

The rest of the proof is similar to that of Theorem 1 and hence is omitted.

Next, we present an interesting result, where condition on ψ is weakened, i.e., we replace condition (3.1) by the following one.

$$\psi(x) \geq c > 0 \qquad \text{for all} \quad x . \tag{3.4}$$

The result is an immediate consequence of Theorem 3, so we omit the proof.

THEOREM 4. Let conditions (2.1), (2.3), (3.2) and (3.4) hold and assume that there exists a function $\rho \in C^2[t_0, \infty), (0, \infty)]$ such that

$$\ddot{\rho}(t) \leq 0 \quad \text{and} \quad \dot{\rho}(t) + a(t)\left(\frac{\rho(t)}{a(t)}\right)^{\cdot} \leq 0 \quad \text{for} \quad t \geq t_0 . \tag{3.5}$$

If condition (2.5) holds, then equation (1.2) is oscillatory.

The following examples are illustrative.

EXAMPLE 5. Consider the differential equation

$$\left(\frac{1}{t}e^{|x|}\dot{x}\right)^{\cdot} + \frac{2}{t^2}\dot{x} + \frac{1}{t^2 \ln t}\left(1 - \frac{2}{t}\right) x = 0 , \quad \text{for} \quad t \geq t_0 = e . \tag{3.6}$$

The conditions (2.1), (2.3), (2.5) and (3.2) of Theorem 3 are satisfied for $\rho(t) = t$. The upper bound c_1 of the function $e^{|x|}$ is undefined and hence both conditions (3.1) and (3.3) fail. Equation (3.6) has a nonoscillatory solution $x(t) = \ln t$.

EXAMPLE 6. Consider the differential equation

$$(t\, e^{|x|}\dot{x})^{\cdot} + \frac{1}{\sqrt{t}}\,|x[\beta t^{\beta_1} \pm \beta_2 \sin t]|^{\alpha} \operatorname{sgn} x[\beta t^{\beta_1} \pm \beta_2 \sin t] = 0 , \tag{3.7}$$

where $\beta > 0$, $\beta_1 > 0$ and $\beta_2 \geq 0$, $\alpha > 0$ and $t \geq t_0 = 1$. We take $\rho(t) = \sqrt{t}$ and $\sigma(t) = \beta t^{\beta_1} - \beta_2$. All the conditions of Theorem 4 are satisfied and so, every solution of equation (3.7) is oscillatory.

We note that the results in [1] - [9] cannot be applied to equation (3.7) since, some of the conditions of the form

$$\frac{f'(x)}{\psi(x)} \geq K > 0 \quad \text{for } x \neq 0 , \quad \text{or} \int^{\pm\infty} \frac{\psi(u)}{f(u)}du < \infty , \quad \text{or} \int_{\pm 0} \frac{\psi(u)}{f(u)}du < \infty ,$$

required in these papers, are not satisfied.

EXAMPLE 7. Consider the differential equation

$$((2 - \sin x)\dot{x})^{\cdot} + \frac{1}{t}\dot{x} + t^{-7/6}|x[g(t)]|^{\alpha} \operatorname{sgn} x[g(t)] = 0 , \; t \geq t_0 = 1 , \tag{3.8}$$

where $\alpha > 0$ and $g(t)$ satisfies either (i) or (ii):

(i) g is a nondecreasing continuous function for $t \geq t_0$ with $\lim_{t\to\infty} g(t) = \infty$.

(ii) $g(t) = \beta t^{\beta_1} \pm \beta_2 \cos t$, $\beta > 0$, $\beta_1 > 0$ and $\beta_2 \geq 0$.

We let $\sigma(t) = g(t)$ in case (i) and $\sigma(t) = \beta t^{\beta_1} - \beta_2$ in case (ii) and take $\rho(t) = t^{1/6}$.

The conditions of Theorem 3 are satisfied and so, every solution of equation (3.8) is oscillatory.

It is easy to check that Theorem 4 is not applicable to equation (3.8) because condition (3.5) is violated.

Next, we consider the differential equation

$$((1 + x^2)\dot{x})^{\cdot} + \frac{1}{t}\,\dot{x} + \frac{1}{t}\,|x[g(t)]|^{\alpha} \operatorname{sgn} x[g(t)] = 0 , \quad t \geq t_0 = 1 , \tag{3.9}$$

where $\alpha > 0$ and $g(t)$ is as in equation (3.8). Equation (3.9) is oscillatory by Theorem 4 for $\rho(t) = 1$.

It is easy to verify that Theorem 3 fails to apply to equation (3.9), since condition (3.1) is not satisfied.

REMARK. The above examples illustrate that our results apply to superlinear, linear or sublinear damped differential equations. Moreover, since we impose no restrictions on the function g in equations (1.1) and (1.2), our results are applicable to ordinary, retarded, advanced and equations of mixed type.

We believe that the oscillatory behavior of equation (3.7) - (3.9) is not deducible from any other known oscillation criteria.

Finally, we give the following oscillation criterion which is similar to the one in Theorem 2. Here we omit the proof.

THEOREM 5. Let conditions (2.1), (2.3), (3.1) and (3.2) hold, $\sigma(t) \geq t$ for $t \geq t_0$, and

$$\int^{\infty} \frac{\psi(u)}{f(u)} du < \infty \qquad \text{and} \qquad \int^{-\infty} \frac{\psi(u)}{f(u)} du < \infty . \tag{3.10}$$

Assume that there exists a function $\rho \in C^2[[t_0,\infty), (0,\infty)]$ such that

$$\ddot{\rho}(t) \geq 0 , \quad \gamma(t, c_1) \geq 0 , \quad \dot{\gamma}(t, c_1) \leq 0 \quad \text{for} \quad t \geq t_0 \tag{3.11}$$

and

$$\limsup_{t \to \infty} \frac{t}{\rho(t)} < \infty .$$

If condition (2.15) holds, than equation (1.2) is oscillatory.

The following example is illustrative.

EXAMPLE 8. Consider the differential equation

$$((2 - \sin x)\dot{x})^{\cdot} + \frac{1}{t}\dot{x} + \frac{1}{t^2}|x[g(t)]|^{\alpha}\operatorname{sgn} x[g(t)] = 0 , \quad t \geq t_0 > 0 ,$$

where $\alpha > 1$ and $g(t)$ is any nondecreasing continuous function with $g(t) \geq t$ for $t \geq t_0$. It is easy to check that the conditions of Theorem 5 are satisfied with $\rho(t) = t$ and hence equations (3.12) is oscillatory.

REMARKS.

1. If $p(t) = 0$, then conditions (2.2) and (3.2) take the form

$$\int^{\infty} \frac{1}{a(s)} ds = \infty ,$$

and condition (3.2) can be replaced by condition (3.4).

2. The results of this paper can be applied to equations of the form (1.1) and (1.2) when f is not a monotonic function. In that case, we can introduce a continuous, nondecreasing function F on R such that

$$xF(x) > 0 \quad \text{and} \quad \frac{f(x)}{F(x)} \geq \delta > 0 \qquad \text{for} \quad x \neq 0 . \tag{3.13}$$

For illustration we can consider the following differential equation.

$$(t\psi(x)\dot{x})^{\cdot} + t^{-\epsilon}f(x[g(t)]) = 0 , \quad t \geq t_0 > 0 , \quad 0 \leq \epsilon \leq \frac{1}{2} , \tag{3.14}$$

where g is as in Example 7, f is any continuous, nondecreasing function on R

with $xf(x) > 0$ for $x \neq 0$ e.g. $f(x) = |x|^{\alpha}$ sgnx, $\alpha > 0$ or $f(x) = \sinh x$ or f is any continuous function on R satisfying condition (3.13) e.g. $f(x) = |x|^{\alpha} e^{\sin x}$sgnx, $\alpha > 0$...etc. and ψ is any continuous function on R satisfying condition (3.4) e.g. $\psi(x) = 1 + x^2$ or e^x or $\ln(e + x^2)$ or $2 \pm \sin x$. If we take $\rho(t) = \sqrt{t}$, the conditions of Theorem 4 are satisfied and thus all solutions of equation (3.14) are oscillatory.

In the case $\psi(x) = 1$, $\varepsilon = 0$, $f(x) = x$ and $g(t) = t$, equation (3.14) has the oscillatory solution $x(t) = \sin\ln t$.

3. The results of this paper are presented in a form which is essentially new.

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