ASYMPTOTIC BEHAVIOR OF RETARDED DIFFERENTIAL EQUATIONS

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ABSTRACT. Some integral criteria for the asymptotic behavior of oscillatory solutions of higher order retarded differential equations are given.

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1. INTRODUCTION.

Recently, Tong [1] proved the following interesting result.

Theorem. Let $f(t, u)$ be continuous on $\mathbb{R}_+ \times \mathbb{R}$. If there are two non-negative continuous functions $v(t)$, $p(t)$ for $t \geq 0$, and a continuous function $g(u)$ for $u \geq 0$ such that

$$(a) \int_1^\infty v(t)p(t)dt < \infty.$$

$$(b) g(u) \text{ is positive and nondecreasing for } u > 0,$$

$$(c) |f(t, u)| \leq v(t)p(t)g(t^{-1}|u|) \text{ for } t \geq 1, u \in \mathbb{R},$$

then the equation

$$u'' + f(t, u) = 0$$

has solutions which are asymptotic to $a+bt$, where a, b are constant and $b \neq 0$.

In this note we generalize Tong's result to a more general case which improves also the results of Chen and Yeh [2] and Kusano and Singh [3]. Using this result, we establish an asymptotic behavior of oscillatory solutions of retarded differential equations.

2. MAIN RESULTS.

Consider the following retarded differential equations

$$(2.1) \quad L_n y(t) + f(t, y(g(t))) = h(t), \quad t \geq 0, \quad n \geq 2$$

where L_n is an operator defined by

$$L_0 y(t) := \frac{y(t)}{r_0(t)}, \quad L_i y(t) := \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1} y(t), \quad i = 1, 2, \dots, n$$

$$r_n(t) := 1.$$

Here $r_i(t) \in C^{n-i}[R_+, R]$ with $r_i(t) > 0$ for $i = 0, 1, \dots, n-1$.

Sufficient smoothness to guarantee the existence of solutions of (2.1) on an infinite subinterval of R_+ will be assumed without mention. The following conditions are assumed to hold in this note.

(i) $f \in C[R_+ \times R, R]$ and there exist two positive functions $p(t)$, $H(t) \in C[R_+, R_+]$ with $H(t)$ nondecreasing and $kH(t) \leq H(kt)$ for any $k > 0$ such that

$$|f(t, u)| \leq p(t)H(|u|),$$

$$(ii) \quad g, h \in C[R_+, R], \quad g(t) \leq t, \quad \lim_{t \rightarrow \infty} g(t) = \infty,$$

$$(iii) \quad \liminf_{t \rightarrow \infty} \frac{1}{r_0(t)} > 0, \quad \limsup_{t \rightarrow \infty} \frac{w_i(t, u)}{w_{n-1}(t, u)} < \infty, \quad i = 1, 2, \dots, n-2,$$

where $w_i(t, u)$ is defined by

$$w_i(t, u) := \int_u^t r_1(s_1) \int_u^{s_1} r_2(s_2) \cdots \int_u^{s_{i-1}} r_i(s_i) ds_i \cdots ds_2 ds_1.$$

Theorem 1. Let

$$(2.2) \quad \int_0^\infty w_{n-1}(t) p(t) dt < \infty$$

$$(2.3) \quad \int_0^\infty |h(t)| dt < \infty$$

hold. If $y(t)$ is a solution of (2.1), then $y(g(t)) = O(w_{n-1}(t, T))$ for some $T \geq 0$.

Proof. Let $y(t)$ be a solution of (2.1) on an interval $[T_0, \infty)$, $T_0 \geq 0$. It follows from (ii) and (iii) that there exist a $T \geq T_0$ and a positive constant m such that

$$g(t) \geq T_0 \quad \text{for } t \geq T$$

and

$$\inf_{t \geq T} \frac{1}{r_0(t)} = \frac{1}{m}.$$

By (iii), there is a positive constant c such that

$$w_i(t, T) \leq c w_{n-1}(t, T), \quad i = 1, 2, \dots, n-2.$$

Now a simple argument shows that

$$\frac{|y(g(t))|}{m} \leq |L_0 y(g(t))| \leq \sum_{i=0}^{n-1} |L_i y(T)| w_i(g(t), T)$$

$$+ \int_T^{g(t)} r_1(s_1) \int_T^{s_1} r_2(s_2) \cdots \int_T^{s_{n-2}} r_{n-1}(s_{n-1}) \int_T^{s_{n-1}} |L_n y(s)| ds ds_{n-1} \cdots ds_1$$

$$\leq c w_{n-1}(t, T) \sum_{i=0}^{n-1} |L_i y(T)| + w_{n-1}(t, T) \int_T^t |L_n y(s)| ds.$$

Hence

$$\begin{aligned} \frac{|y(g(t))|}{w_{n-1}(t, T)} &\leq c m \sum_{i=0}^{n-1} |L_i y(T)| + m \int_T^t |h(s)| ds + m \int_T^t p(s) H(y(g(s))) ds \\ &\leq M + m \int_T^t w_{n-1}(s, T) p(s) H\left(\frac{|y(g(s))|}{w_{n-1}(s, T)}\right) ds, \end{aligned}$$

where

$$M := c m \sum_{i=0}^{n-1} |L_i y(T)| + m \int_T^\infty |h(s)| ds.$$

By Bihari's inequality [4] or LaSalle's inequality [5] we have

$$\frac{|y(g(t))|}{w_{n-1}(t, T)} \leq G^{-1}\left(G(M) + \int_T^t w_{n-1}(s, T) p(s) ds\right),$$

where $G(x) := \int_T^x \frac{dt}{H(t)}$ and $G^{-1}(x)$ is the inverse function of $G(x)$. This and

(2.2) imply $\frac{|y(g(t))|}{w_{n-1}(t, T)}$ is bounded. This completes the proof.

Remark 1. For $n = 2$, $r_0(t) = r_1(t) = 1$ and $g(t) = t$, Theorem 1 improves Tong's result [1].

Remark 2. For $H(u) = |u|^r$, where $r \in (0, 1]$, Theorem 1 improves the results of Chen and Yeh [2, Theorem 1] and Singh and Kusano [3, Theorem 1] which require the condition

$$\int_0^\infty r_i(t) dt = \infty, \quad \text{for } i = 1, 2, \dots, n-1.$$

Using Theorem 1, we can prove the following theorem which extends Theorem 3 of Philos [6].

Theorem 2. Let (2.2) and (2.3) hold. Assume that for some $T \geq 0$

$$(2.4) \quad \int_T^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \int_{s_{n-1}}^\infty p(s) H(c w_{n-1}(s, T)) ds ds_{n-1} \cdots ds_1 < \infty$$

for any constant $c > 0$, and

$$(2.5) \quad \int_T^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \int_{s_{n-1}}^\infty |h(s)| ds ds_{n-1} \cdots ds_1 < \infty$$

hold. Then every oscillatory solution $y(t)$ of (2.1) satisfies

$$\lim_{t \rightarrow \infty} L_i y(t) = 0 \quad \text{for } i = 1, 2, \dots, n-1.$$

The proof of Theorem 2 is essentially the same as that of Theorem 3 in [6], so we omit the details.

Example 1. The differential equation

$$(ty'(t))' + \frac{1}{t} y(t) = \frac{2}{t^2}, \quad t \geq 1$$

has an oscillatory solution $y(t) = \frac{1}{t} \sin(1nt)$ but $\lim_{t \rightarrow \infty} y(t)$ does not exist.

In this example, condition (2.2) and (2.4) are not satisfied, while (2.3) and (2.5) are valid.

Example 2. Consider the differential equation

$$(e^{-t}y')' + e^{-3t-\pi}y(t-\pi) = e^{-2t}[\sin t + 7\cos t - e^{-2t}\sin t],$$

for $t \geq 0$. All conditions of Theorem 2 are satisfied. It has $y(t) = e^{-t}\sin t$ as an oscillatory solution which approaches zero as $t \rightarrow \infty$.

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