Internat. J. Math. & Math. Sci.
Vol. 10, No.3 (1987) 621-623

M-QUASI-HYPONORMAL COMPOSITION OPERATORS

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(Received November 12, 1985)

ABSTRACT. A necessary and sufficient condition is obtained for M-quasi-hyponormal composition operators. It has also been proved that the class of M-quasi-hyponormal composition operators coincides with the class of M-paranormal composition operators. Existence of M-hyponormal composition operators which are not hyponormal; and M-quasi-hyponormal composition operators which are not M-hyponormal and quasi-hyponormal are also shown.

KEY WORDS AND PHRASES. M-hyponormal, M-quasi-hyponormal, M-paranormal, normal composition operators.
1980 AMS SUBJECT CLASSIFICATION CODE. 47

1. PRELIMINARIES.

Let (X,S,m) be a sigma-finite measure space and T a measurable transformation from X into itself (that is one $mT^{-1}(E) = 0$ whenever $m(E) = 0$ for $E \in S$). Then the equation $C_T f = f o T$ for every f in $L^2(m)$ defines a linear transformation. If C_T is bounded with range in $L^2(m)$, then it is called composition operator. If X = N the set of all non-zero positive integers and m is counting measure on the family of all subsets of N, then $L^2(m) = \ell^2$ (the Hilbert space of all square summable sequences).

Let $f_o = \frac{dmT^{-1}}{dm}$ be the Radon-Nikodym derivative of the measure mT^{-1} with respect to the measure m,

$$\frac{dm(ToT)^{-1}}{dmT^{-1}} = g_o, \qquad \frac{dm(ToT)^{-1}}{dm} = h_o$$

Then $h_o = f_o \cdot g_o$.

Let B(H) denote the Banach algebra of all bounded linear operators on the Hilbert space H. An operator $T \in B(H)$ is called M-quasi-hypornormal if there exists $M > 0$ such that

$$M^2 T^{*2} T^2 - (T^* T)^2 \geq 0$$

or equivalently $||T^*Tx|| \leq M\, ||T^2x||$ for all x in H [1]. T is said to be M-paranormal [2] if for all unit vectors x in H

$$||Tx||^2 \leq M\, ||T^2x||.$$

T is said to be M-hyponormal [2] if

$$||T^*x|| \leq M\, ||Tx|| \quad \text{for all x in H.}$$

The purpose of this paper is to generalize the results on quasi-hyponormal composition operators in [3] for M-quasi-hyponormal composition operators.

2. M-QUASI-HYPONORMAL COMPOSITION OPERATORS.

In this section we obtain a necessary and sufficient condition for M-quasi-hyponormal composition operators and then show that the class of M-quasi-hyponormal composition operators on ℓ^2 coincides with the class of M-paranormal composition operators. We also show the existence of M-hyponormal composition operators which are not hyponormal, and M-quasi-hyponormal composition operators which are not M-hyponormal and quasi-hyponormal.

THEOREM 2.1. Let $C_T \in B(L^2)$. Then C_T is M-quasi-hyponormal if and only if $f_o^2 \leq M^2 h_o$.

PROOF. Since for any f in L^2,

$$(C_T^{*2} C_T^2 f,f) = (C_T^2 f, C_T^2 f) = \int h_o\, |f|^2\, dm,$$

$$= (M_{h_o} f,f),$$

where M_{h_o} is the multiplication operator induced by h_o, therefore $C_T^{*2} C_T^2 = M_{h_o}$. Similarly it can be seen that $C_T^* C_T = M_{f_o}$. C_T is M-quasi-hyponormal if and only if

$$M^2 C_T^{*2} C_T^2 - (C_T^* C_T)^2 \geq 0.$$

This implies that

$$M^2 M_{h_o} - M_{f_o}^2 \geq 0,$$

that is $f_o^2 \leq M^2 h_o$.

Hence the result.

COROLLARY. Let $C_T \in B(\ell^2)$. Then C_T is M-quasi-hyponormal if and only if $f_o \leq M^2 g_o$.

PROOF. Since $h_o = f_o \cdot g_o$ and f_o is positive, therefore, by above theorem we get the result.

THEOREM 2.2. Let $C_T \in B(\ell^2)$. Then C_T is M-quasi-hyponormal if and only if C_T is M-paranormal.

PROOF. Necessity is true for any bounded operator A. For the sufficiency, let C_T be M-paranormal, then

$$||C_T X_{\{n\}}||^2 \leqq M||C_T^2 X_{\{n\}}|| \quad \text{for all } n \in N$$

$$\text{or } \int |X_{\{n\}} oT|^2 dm \leqq M(\int |X_{\{n\}} oT^2|^2 dm)^{1/2}$$

$$\text{or } \int |X_{\{n\}}|^2 dmT^{-1} \leqq M(\int |X_{\{n\}}|^2 dm(ToT)^{-1})^{1/2}$$

$$\text{or } \int_{\{n\}} f_o \, dm \leqq M(\int_{\{n\}} h_o \, dm)^{1/2}$$

$$\text{or } f_o^2 (n) \leqq M^2 h_o (n) \quad \text{for all n in N.}$$

Hence $f_o^2 \leqq M^2 h_o$; C_T is M-quasi-hyponormal.

THEOREM 2.3. Let $C_T \in B(\ell^2)$ and $T:N \to N$ be one-to-one. Then the following are equivalent.

(i) Normal

(ii) M-hyponormal

(iii) M-quasi-hyponormal.

PROOF. (i) implies (ii), (ii) implies (iii) are always true for any bounded operator A. We show that (iii) implies (i). Let C_T be M-quasi-hyponormal. Then $||C_T^* C_T f|| \leqq M \, ||C_T^2 f||$ for all f in ℓ^2. Now T is onto because if T is not onto then $N|T(N)$ is non-empty and for $n \in N|T(N)$

$$||C_T^* C_T X_{\{n\}}|| = 1 \text{ and } ||C_T C_T X_{\{n\}}|| = 0.$$

There exists no M>0 such that C_T is M-quasi-hyponormal which is a contradiction. Since T is one-to-one, therefore, T is invertible, by Theorem 2.2 [4] C_T is invertible and C_T is normal by Theorem 2.1 [3].

Here we give an example of a composition operator on ℓ^2 which is M-hyponormal but not hyponormal.

EXAMPLE 1. Let T:N N be the mapping such that

$T(1) = 2$, $T(2) = 1$, $T(3) = 2$ and

$T(3n+m) = n+2$, $m = 1,2,3$ and $n \in N$.

Then C_T is not hyponormal as $f_o oT \nleq f_o$ for n = 1. C_T is M-hyponormal for $M \geq \sqrt{2}$.

EXAMPLE 2. Let $T:N \to N$ be defined by $T(1) = 2$, $T(2) = 1$, $T(3n+m) = n+1$, $m = 0,1,2$ and $n \in N$. Then C_T is $\sqrt{2}$ - quasi-hyponormal but C_T is not $\sqrt{2}$ -hyponormal. C_T is not quasi-hyponormal also.

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