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FINITE p'-NILPOTENT GROUPS. I

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ABSTRACT: In this paper we consider finite p'-nilpotent groups which is a generalization of finite p-nilpotent groups. This generalization leads us to consider the various special subgroups such as the Frattini subgroup, Fitting subgroup, and the hypercenter in this generalized setting. The paper also considers the conditions under which product of p'-nilpotent groups will be a p'-nilpotent group.

KEY WORDS AND PHRASES. *Frattini subgroup, nilpotent group, solvable group, hypercenter, maximal subgroup, saturated formation*
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1. INTRODUCTION.

We consider only finite groups. It is well known that a group is p-nilpotent if it has a normal complement. We generalize this concept by defining a group G to be π-nilpotent, π a set of primes, if G has a normal π'-subgroup N with G/N a nilpotent π-group. Let $\underline{P}$ be the set of all primes. When $\pi = \{p\}$, π-nilpotency is same as p-nilpotency. When $\pi = \underline{P} - \{p\}$, π-nilpotency is called p'-nilpotency. In 1959 W.E. Deskins [1] defined the p-Frattini subgroup, $\Phi_p(G)$, as the intersection of all maximal subgroups of p-free index in G. He showed that $\Phi_p(G)$ is p'-nilpotent [2]. M. Torres [3] defined $\Phi^*(G) = \bigcap_{p\in\underline{P}} \Phi_p(G)$. Results similar to those for $\Phi(G)$ were obtained by E. Arrington-Idowu [4] for $\Phi_p(G)$ and by M. Torres for $\Phi^*(G)$. We use these results and obtain characterizations for a group to be nilpotent, metanilpotent. Using known results on p-nilpotent groups we observe that p'-nilpotent groups form a saturated formation $\underline{F}_p$. We obtain results on the $\underline{F}_p$-hypercenter of G similar to those known for the usual hypercenter of G and also a characterization for a group to be p'-nilpotent. Some additional results are also proved. We use standard notation and terminology as in [5].

2. DEFINITIONS AND KNOWN RESULTS.

DEFINITION 2.1 : G is π-nilpotent, π a set of primes, if G_π, $\lhd$ G and G/G_π, is a nilpotent π-group. When $\pi = \underline{P} - \{p\}$, G is called a p'-nilpotent group.

EXAMPLE 2.2 : Let $G = A_5 \times H$ where H is nilpotent and 2,3,5 do not lie in $\pi(H)$. G is π-nilpotent for $\pi = \pi(H)$. G is not solvable.

Thus, a π-nilpotent group need not be solvable in general. However, a p'-nilpotent group is always solvable.

The following proposition is easy to prove.

PROPOSITION 2.3 : G is π-nilpotent if and only if G is p-nilpotent $\forall\ p \in \pi$.

COROLLARY 2.4 : G is p'-nilpotent if and only if G is q-nilpotent $\forall\ q \neq p$.

It is well known that p-nilpotent groups form a subgroup closed saturated formation and that the intersection of two subgroup closed saturated formations is a subgroup closed saturated formation. In view of Corollary 2.4 we then have that the p'-nilpotent groups form a subgroup closed saturated formation, $\underline{F}_p$. We define $\underline{F}_p$ locally as follows in order to make it integrated.

$$\underline{F}_p(p) = \{ \text{ all p'-nilpotent groups } \}$$

$$\underline{F}_p(q) = \{1\} \quad \forall\ q \neq p.$$

DEFINITION 2.5 : The $\underline{F}_p$-<u>hypercenter</u> of G , $Z_{\underline{F}_p}(G)$, is the largest normal subgroup of G all of whose G-chief factors are $\underline{F}_p$-central.

DEFINITION 2.6 : Let $\underline{F}$ be a formation having an integrated local definition. $N \leq G$ is called an $\underline{F}$-<u>immersed</u> subgroup of G if: (i) $N \lhd G$, (ii) all G-chief factors that lie in N are $\underline{F}$-central.

DEFINITION 2.7 : A formation $\underline{F}$ is said to be <u>normally closed</u> if $G \in F$ and $N \lhd G$, then $N \in \underline{F}$.

Using the following theorem of M. Hale we can conclude that $Z_{\underline{F}_p}(G)$ is p'-nilpotent.

THEOREM 2.8 (M. Hale, Prop. 6 of [6]) : For a saturated formation $\underline{F}$, $\underline{F}$-immersed subgroups lie in $\underline{F}$ if and only if $\underline{F}$ is normally closed.

We include the following two theorems for easy reference.

THEOREM 2.9 (E. Arrington-Idowu) : Let G be a group.

(i) $x \in \Phi_p(G)$ if and only if $G = \langle R , x \rangle$ with $p \nmid [G : \langle R \rangle]$ implies $G = \langle R \rangle$. (1.1.3 of [4]).

(ii) $M \lhd G$ implies $\Phi_p(M) \leq \Phi_p(G)$. (1.1.7 of [4]).

(iii) $\Phi_p(G) = G$ if and only if G is a p-group. (1.1.2 of [4]).

(iv) if G is p'-nilpotent, then every maximal subgroup of p-free index is normal in G. (2.1.10 of [4]).

(v) $F_p(G/\Phi_p(G)) = F_p(G)/\Phi_p(G)$, where $F_p(G)$ is the largest normal p'-nilpotent subgroup of G. (2.2.3 of [4]).

(vi) let D and M be normal subgroups of G with $D \leq M \cap \Phi_p(G)$. Then M is p'-nilpotent if and only if M/D is p'-nilpotent. (2.1.7 of [4]).

THEOREM 2.10 (M. Torres [3]) : $\Phi^*(G)/F(G) \leq \Phi(G/F(G))$.

It is easy to verify that the product of normal p'-nilpotent subgroups of G is a normal p'-nilpotent subgroup of G. Thus, every group G possesses a unique largest normal p'-nilpotent subgroup, $F_p(G)$.

DEFINITION 2.11 : $F^*(G) = \prod_{p \in \pi} F_p(G)$.

It is easy to see that $O_p(G)$ is the Sylow p-subgroup of $F_p(G)$ and $\Phi_p(G)$. In the light of this observation the following inclusions are obvious:

$$\Phi(G) \leq F(G) \leq \Phi^*(G) \leq F^*(G).$$

3. $\underline{F^*(G),\ \Phi^*(G)}$.

LEMMA 3.1 : $F_p(G)/O_p(G) = F(G/O_p(G))$.

PROOF: $F_p(G)$ is p'-nilpotent and the Sylow p-subgroup of $F_p(G)$ is $O_p(G)$. Thus $F_p(G)/O_p(G) \leqq F(G/O_p(G)) = N/O_p(G)$, say. Since $(N/O_p(G))_p = N_p/O_/(G)$ char $N/O_p(G) \lhd G/O_p(G)$ implies $N_p/O_p(G) \lhd G/O_p(G)$, we have $N_p \lhd G$. Hence $N_p = O_p(G)$. Therefore, $N/O_p(G)$ is a nilpotent group of p-free order and hence N is a p'-nilpotent normal subgroup of G. Thus $N \leqq F_p(G)$. This shows that $F_p(G)/O_p(G) = F(G/O_p(G))$. Q.E.D.

THEOREM 3.2 : $F^*(G)$ and $\Phi^*(G)$ are metanilpotent.

PROOF: $F_p(G)/F(G) \simeq (F_p(G)/O_p(G))/(F(G)/O_p(G))$ shows that $F_p(G)/F(G)$ is nilpotent. Hence $\prod_{p\in P} (F_p(G)/F(G)) = (\prod_{p\in \underline{P}} F_p(G))/F(G)$ is nilpotent, i.e., $F^*(G)/F(G)$ is nilpotent. Hence $F^*(G)$ is metanilpotent. Since $\Phi^*(G) \leqq F^*(G)$, $\Phi^*(G)$ is also metanilpotent. Q.E.D.

PROPOSITION 3.3 : (i) $F_p(G/\Phi(G)) = F_p(G)/\Phi(G)$,

(ii) $F^*(G/\Phi(G)) = F^*(G)/\Phi(G)$.

PROOF: $F_p(G)/\Phi(G)$ is a p'-nilpotent normal subgroup of $G/\Phi(G)$. Hence $F_p(G)/\Phi(G) \leqq F_p(G/\Phi(G))$. Let $F_p(G/\Phi(G)) = N/\Phi(G)$. $(N/\Phi(G))_p = N_p\Phi(G)/\Phi(G)$ char $N/\Phi(G) \lhd G/\Phi(G)$ implies $N_p\Phi(G)/\Phi(G) \lhd G/\Phi(G)$ and hence $N_p\Phi(G) \lhd G$. Using Frattini argument, we have $G = N_G(N_p)\Phi(G)$. Hence $G = N_G(N_p)$. Thus $N_p \lhd G$. Moreover, $N/N_p\Phi(G) \simeq (N/\Phi(G))/(N_p\Phi(G)/\Phi(G))$ is nilpotent. Therefore, $N_qN_p\Phi(G) \lhd G$. Using the generalized Frattini argument we have $G = N_G(N_qN_p)\Phi(G)$. Hence $G = N_G(N_qN_p)$. Thus $N_qN_p \lhd G\ \forall q$. Since N is solvable N can be written as a permutable product of its Sylow subgroups, say, $N = N_{p_1} \ldots N_{p_r}$. Take $N_{p_1} = N_p$. Using the previous argument, we have $N_{p_1}N_{p_i} \leqq F_p(G)\ \forall\ i$. Thus $N \leqq F_p(G)$ and so (i) follows.

$$F^*(G/\Phi(G)) = \prod_{p\in \underline{P}} F_p(G/\Phi(G)) = \prod_{p\in \underline{P}} F_p(G)/\Phi(G) \text{ , using (i)}$$
$$= (\prod_{p\in \underline{P}} F_p(G))/\Phi(G)$$
$$= F^*(G)/\Phi(G).$$

Q.E.D.

It is well known that $\Phi(G) < G$ for a finite group G. We saw in 2.9(iii) that $\Phi_p(G) = G$ if and only if G is a p-group. We now prove a similar result for $\Phi^*(G)$.

THEOREM 3.4 : G is nilpotent if and only if $\Phi^*(G) = G$.

PROOF: G nilpotent implies $F(G) = G$ and hence $\Phi^*(G) = G$.

Suppose $\Phi^*(G) = G$. We first consider the case $\Phi(G) \neq 1$. In this case consider $G/\Phi(G)$. $\Phi^*(G/\Phi(G)) = \prod_{p\in \underline{P}} \Phi_p(G/\Phi(G))$

$$= \prod_{p\in \underline{P}} (\Phi_p(G)/\Phi(G))$$
$$= (\prod_{p\in \underline{P}} \Phi_p(G))/\Phi(G)$$

$$= \Phi^*(G)/\Phi(G)$$
$$= G/\Phi(G).$$

By induction on $|G|$, $G/\Phi(G)$ is nilpotent and hence G is nilpotent. Next consider the case $\Phi(G) = 1$. If $\Phi^*(G) = \Phi_p(G)$ for some prime p, then $G = \Phi_p(G)$, a p-group by 2.9(iii). Thus G is nilpotent in this case also.

We now assume that $\Phi_p(G) < \Phi^*(G)$ $\forall\, p \in \underline{P}$. Consider $G/O_p(G)$, $G/O_q(G)$ for $p \neq q$.

$$\Phi^*(G/O_p(G)) = \Phi^*(G)/O_p(G) = G/O_p(G).$$
$$\Phi^*(G/O_q(G)) = \Phi^*(G)/O_q(G) = G/O_q(G).$$

By induction on $|G|$, $G/O_p(G)$ and $G/O_q(G)$ are nilpotent. Hence $G \simeq G/(O_p(G) \cap O_q(G)) \hookrightarrow (G/O_p(G)) \times (G/O_q(G))$ implies that G is nilpotent. Q.E.D.

It is well known that G is nilpotent if and only if $G' \leq \Phi(G)$. We now obtain a similar characterization for a group to be metanilpotent, i.e., Fitting length at most 2. First we prove the following lemma.

LEMMA 3.5 : Let $H \lhd G$. Then $H/H \cap \Phi^*(G)$ nilpotent implies that H is metanilpotent.

PROOF: From 2.10 $\Phi^*(G)/F(G) \leq \Phi(G/F(G))$. Let $\Phi(G/F(G)) = X/F(G)$. $H\Phi^*(G)/\Phi^*(G) \simeq H/H \cap \Phi^*(G)$ is nilpotent by hypothesis. Hence $HX/\Phi^*(G) = (H\Phi^*(G)/\Phi^*(G))(X/\Phi^*(G))$ is nilpotent. Thus $(HX/F(G))/(\Phi^*(G)/F(G)) \simeq HX/\Phi^*(G)$ is nilpotent. Now $(H/F(G))(X/F(G))/(X/F(G)) \simeq \{(H/F(G))(X/F(G))/(\Phi^*(G)/F(G))\} \div \{(X/F(G)/(\Phi^*(G)/F(G))\}$ shows that $(HX/F(G))/(X/F(G))$ nilpotent. Since product of nilpotent normal subgroups is a nilpotent normal subgroup, we see that $H/F(G)$ is a nilpotent normal subgroup of $G/F(G)$, i.e., H is metanilpotent. Q.E.D.

THEOREM 3.6 : $\ell(G) \leq 2$ if and only if $G' \leq \Phi^*(G)$.

PROOF: $G' \leq \Phi^*(G)$ implies $G/\Phi^*(G)$ abelian. Thus G is metanilpotent by 3.5, i.e., $\ell(G) \leq 2$.

Conversely, $\ell(G) \leq 2$ implies that G is solvable. Hence $O_p(G) \neq 1$ for some p. Clearly $\ell(G/O_p(G)) \leq 2$. By induction on $|G|$, $(G/O_p(G))' \leq \Phi^*(G/O_p(G))$. i.e., $G'O_p(G)/O_p(G) \leq \Phi^*(G)/O_p(G)$. Hence $G' \leq G'O_p(G) \leq \Phi^*(G)$. Q.E.D.

4. p'-NILPOTENT GROUPS.

In this section we obtain several results on p'-nilpotent groups. We know that a minimal normal subgroup of a nilpotent group lies in the center of the group. The corresponding result is not true for p'-nilpotent groups, in general, as A_4 shows with p = 2. In the light of this observation we give the following proposition.

PROPOSITION 4.1 : Let G be p'-nilpotent and let N be a minimal normal subgroup of p-free order in G. Then $N \leq Z(G)$.

PROOF: Since G is p'-nilpotent, it is solvable. N is of p-free order implies that $N \leq G^p$ $\forall G^p$. G^p is nilpotent since G is p'-nilpotent. N is a prime power group since G is solvable. N is of p-free order shows that N is a q-group, $q \neq p$.

$G_p \lhd G$ since G is p'-nilpotent. Hence $[N, G_p] = 1$, i.e., $G_p \leqq C_G(N)$. $N \lhd G$ implies $N \leqq G_q$ $\forall$ G_q. This shows that $L = N \cap Z(G_q) \neq 1$. G^p is nilpotent, so $G^p \leqq C_G(L)$. Thus $G = G_pG^p \leqq C_G(L)$, i.e., $C_G(L) = G$. Hence $L = N = N \cap Z(G_q)$ because N is a minimal normal subgroup of G. Hence $N \leqq Z(G_q)$. Combining this with $N \leqq G^p$, G^p nilpotent, we have $G^p \leqq C_G(N)$. Thus $G = G_pG^p \leqq C_G(N)$, i.e., $N \leqq Z(G)$. Q.E.D.

Next we obtain some information on maximal subgroups of p-free index in a group which possesses a p'-nilpotent maximal subgroup.

PROPOSITION 4.2 : Let N be a p'-nilpotent maximal subgroup of G. Then for every maximal subgroup M of p-free index in G we have either $M_G \leqq N_G$ or $M \lhd G$.

The proof follows easily from 2.9(iv).

J.G. Thompson showed that if a group has a maximal subgroup which is nilpotent of odd order then G is solvable, in particular, G is nonsimple. We now prove a similar theorem for a group with a p'-nilpotent maximal subgroup under suitable conditions and give examples to show that the conditions are necessary.

THEOREM 4.3 : Let $N < G$, N p'-nilpotent. If (i) $p \mid [G : N]$,

(ii) N is not a 2-group,

then G is a nonsimple group.

PROOF: (1) Suppose $p \mid |N|$. Then $N_p \lhd N$, i.e., $N \leqq N_G(N_p)$. Since $p \mid [G : N]$, $N_p < G_p$ for some G_p. Hence $N_p < N_G(N_p)$. Let $g \in N_G(N_p) - N_p$. Hence $\langle N, g \rangle \leqq N_G(N_p)$, i.e., $N_p \lhd G$, since $N < G$.

(2) $p \nmid |N|$.

Hence N is nilpotent. If N is not a Hall subgroup of G, then there exists a prime $q \mid (|N|, [G : N])$. As in (1) we see that $N_q \lhd G$. So we now assume that N is a Hall subgroup of G. Suppose N is of odd order. Then using Thompson's theorem mentioned above we see that G is nonsimple, hence we assume that N is of even order, by hypothesis N is not a 2-group. Let r be any prime divisor of $|N|$. Then $N_r \lhd N$ and hence $N \leqq N_G(N_r)$. Since $N < G$ we have either $N_G(N_r) = G$ or $N_G(N_r) = N$. If $N_G(N_r) = G$ for some r, then $N_r \lhd G$ and hence G is nonsimple. On the other hand, if $N_G(N_r) = N$ $\forall$ r dividing $|N|$, then G is not simple by a theorem of Wielandt (see Satz 7.3, p. 444 of [5]). Q.E.D.

REMARK : Hypotheses (i) and (ii) are necessary in 4.3. Take $G = A_5$ and $N = A_4$. $N < G$, N is 2'-nilpotent and $[G : N] = 5$. G is simple. Take $G = PSL(2, 31)$ and $N = G_2$. $N < G$, N is nilpotent and G is simple.

We know that if $N \lhd G$, then $\Phi_p(N) \leqq \Phi_p(G)$ by 2.9(ii). Hence $\Phi_p(N) \leqq \Phi_p(G) \cap N$. The question of when equality holds leads to the next result.

THEOREM 4.4 : Let N be a p'-nilpotent normal Hall subgroup of G. Let $N \cap \Phi_p(G)$ be nilpotent. Then $\Phi_p(N) = N \cap \Phi_p(G)$.

PROOF: Let $D = N \cap \Phi_p(G)$. As noted before $\Phi_p(N) \leqq D$. N p'-nilpotent implies $N_p \lhd G$. Also, $N_p \leqq \Phi_p(N) \leqq \Phi_p(G)$. Hence $p \nmid [D : \Phi_p(N)]$, but for some i,

$p_i \mid [\, D : \Phi_p(N) \,]$ where $N = N_p N_{p_1} \ldots N_{p_r}$. Suppose that $p_{j_1}, p_{j_2}, \ldots, p_{j_s}$ are the only primes that do not divide $[\, D : \Phi_p(N) \,]$ besides p, where $\{j_1, \ldots, j_s\}$ $\{1, \ldots, r\}$. Let M be a normal Hall subgroup of N minimal with respect to $(|M|, [D:\Phi_p(N)]) > 1$. Take $M = N_p N_{p_{j_1}} \ldots N_{p_{j_s}} N_{p_i}$ and note that $(\, |M| \,, [\, D : \Phi_p(N) \,]\,) = p^i > 1$, $p_i \neq p$. M has a normal Hall subgroup K such that $M/K \simeq M_{p_i}$ since M is p'-nilpotent. Let $Q_0 = D_{p_i}$. D nilpotent implies Q_0 char $D \lhd G$, so $Q_0 \lhd G$. Since $p_i \mid |M|$ and M is a Hall subgroup in N, $Q_0 \leq M$. $\Phi_p(M_{p_i}) = \Phi(M_{p_i})$. Consider $L = K\Phi(M_{p_i}) \leq M$. Since $L \leq N_M(L)$, $M_{p_i} \leq N_M(K) = M$. $M_{p_i} \leq N_M(\Phi M_{p_i}))$, we have $L \lhd M$. Suppose $Q_0 \leq L$. Since $P_i \nmid |K|$, $Q_0 \leq \Phi(M_{p_i})$. Using Hilfssatz 3.3(a), p.269 of [5], $Q_0 \leq \Phi(N) \leq \Phi_p(N)$; i.e., $p_i \nmid [\, D : \Phi_p(N) \,]$. This is a contradiction and so $Q_0 \not\leq L$. We now show that this too leads to a contradiction. Let $R = LQ_0$. M is a normal Hall subgroup of N so that M is a normal Hall subgroup of G, since N is a normal Hall subgroup of G. Using Schur's complementation theorem (Theorem 2.1, p.221 of [7]) G = MV, M V = 1. Since $L = K\Phi(M_{p_i})$ and $M = KM_{p_i}$, M/L is an elementary abelian p_i- group. Further, $p_i \nmid |V|$. Consider $G/L = (M/L)\cdot(VL/L)$. $VL/L \simeq V$, so V can be considered as operating on a module M/L over $GF(p_i)$. We can apply Maschke's theorem to $R/L \leq M/L$ since $P_i \nmid |V|$. Hence $M/L = (R/L) \times (R_1/L)$ where $R_1/L \lhd G/L$. i.e., $M = RR_1$ and $R \cap R_1 = L$. $Q_0 \not\leq L$ implies that $L < R$, so $R_1 < M$. Hence $R_1V < G$. $R_1V \leq U <\cdot G$ for some $U <\cdot G$. $L \leq R_1 < R_1V \leq U$. $M = KM_{p_i}$ and $p, p_i \in \pi(M)$. Therefore, $G_p \leq K \leq L < U$; i.e., $[\, G : U \,]$ is p-free. Hence $\Phi_p(G) \leq U$. By choice of Q_0, $Q_0 \leq D \leq \Phi_p(G) \leq U$. Therefore, $LQ_0R_1V \leq U$. $LQ_0R_1V = RR_1V = MV = G \leq U$. Thus we arrive at a contradiction when we assume that $\Phi_p(N) < D$. Hence $\Phi_p(N) = D$. Q.E.D.

COROLLARY 4.5 : If F(G) is a Hall subroup of G, then $\Phi_p F(G)) = F(G) \quad \Phi_p(G) \;\; \forall\, p$.

THEOREM 4.6 : Let G be solvable with $M \lhd N \lhd G$ and let N be a Hall subgroup of G with $N \cap \Phi_p(G)$ nilpotent. Let π be a set of primes containing p. Then $N/(M(N \cap \Phi_p(G)))$ π-closed implies N/M π-closed.

PROOF : Let $L = M(N \cap \Phi_p(G))$ and let H/L be the Hall π-subgroup of N/L. $L/M \simeq (N \cap \Phi_p(G))/(M \cap \Phi_p(G))$, a nilpotent group. Hence L/M has a normal Hall '-subgroup K/M and $(L/M)/(K/M) \simeq L/K$, a π-subgroup. K/M char $L/M \lhd H/M$ implies $K/M \lhd H/M$.

(1) We shall show that K/M is a Hall π'-subgroup of H/M. Suppose $q \mid (\,|K/M|\,,\ [H/M : K/M]\,)$. $q \mid |K/M|$ implies q is a π'-number. $q \mid [H/M : K/M] = [H : K]$ implies $q \mid [L : K]$, so q is a π-number. Hence q = 1. Applying Schur's complementation theorem to K/M as a normal Hall subgroup of H/M we have $H/M = (K/M)\cdot(A/M)$ with $K \cap A = M$. Applying generalized Frattini argument, we have $N/M = (N_{N/M}(A/M))\cdot(H/M) = (N_N(A)H)/M$. Hence $N = N_N(A)H = N_N(A)AK = N_N(A)K = N_N(A)L$, since $K \leqq L$.

$= N_N(A)M(N \cap \Phi_p(G))$

$= N_N(A)\ \Phi_p(N)$, since $M \leqq A$ and

$\Phi_p(N) = N \cap \Phi_p(G)$ from 4.4. By hypothesis $p \in \pi$, so $N_N(A)$ has p-free index in N. Applying 2.9(i), we have $N_N(A) = N$, i.e., $A \lhd N$.

(2) We shall show that A/M is a Hall π-subgroup of N/M. $[N/L : H/L] = [N : H] = [N/M : H/M]$ is a π'-number. $[N/M : A/M] = [N/M : H/M]\,[H/M : A/M]$. $[H/M : A/M]$ is a π'-number. Thus we have shown that N/M is π-closed. Q.E.D.

THEOREM 4.7 : Let G be solvable with $M \lhd N \lhd G$ and let N be a Hall subgroup of G with $N \cap \Phi_p(G)$ nilpotent. If $N/(M(N \cap \Phi_p(G)))$ is p'-nilpotent, then N/M is p'-nilpotent.

PROOF : Let $L = M(N \cap \Phi_p(G))$. N/L p'-nilpotent implies N/L p-closed. Hence N/M p-closed by 4.6. N_pM/M char N/M. $N/N_pL \simeq (N/L)/(N_pL/L)$ is nilpotent. Let $q \mid |N/N_pM|$, so $q \neq p$. Also, $q \mid |N/N_pL|$. Take $\pi = \{p\ ,\ q\}$. N/N_pL is π-closed. Apply 4.6 to N_pM and N/N_pL and conclude that N/N_pM is π-closed; i.e., $\forall\ q \mid |N/N_pM|$, N/N_pM has its Sylow q-subgroup normal. Hence N/N_pM is nilpotent; i.e., N/M is p'-nilpotent. Q.E.D.

H. Wielandt has shown that if a group possesses three solvable subgroups of pairwise relatively prime indices, then G is solvable (see Satz 1.9, p.662 of [5]). We now prove the corresponding theorem for p'-nilpotent groups.

THEOREM 4.8 : Let G have three p'-nilpotent subgroups of pairwise relatively prime indices. Then G is p'-nilpotent.

PROOF : Let H_i , i = 1,2,3 be p'-nilpotent with $[G : H_i]$ pairwise relatively prime. Let $D = H_1 \cap H_2$ and let $p \mid |H_1|$. Let P_i be the Sylow p-subgroup of H_i. $(\,[G : H_1]\ ,\ [G : H_2]\,) = 1$ implies $G = H_1H_2$. $[G : H_2] = [H_1 : D]$. p divides only one of $[G : H_2]$, $[G : H_3]$. Without loss of generality assume that $p \nmid [G : H_2]$. Hence $P_2 = G_p \cap H_2$. $P_1 \lhd H_1$ implies $P_1D \leqq H_1$. $[P_1D : D] = [P_1 : P_1 \cap D]$ is a power of p. $[P_1D : D]\,[H_1 : P_1D] = [H_1 : D] = [G : H_2]$ shows that $[P_1D : D] \mid [G : H_2]$, i.e., $p \mid [G : H_2]$. This contradiction shows that $P_1D = D$, i.e., $P_1 \leqq D$. $\forall\ g \in G$, $g = h_1h_2$, $h_i \in H_i$, $P_1{}^g = P_1{}^{h_1h_2} = P_1{}^{h_2} \leqq D^{h_2} \leqq H_2$. Let $N = \langle P_1{}^g;\ g \in G\rangle$. $N \lhd G$. $P_1{}^g = P_1{}^{h_2} \leqq P_2$ implies that N is a p-group. Consider G/N. By induction on $|G|$, we have G/N p'-nilpotent, so $G_p/N \lhd G/N$. Hence $G_p \lhd G$. Consider G/G_p and use induction on $|G|$. Hence G/G_p is a p-free order p'-nilpotent group

and hence G/G_p is nilpotent. Therefore, G is p'-nilpotent. Q.E.D.

5. $\underline{G}_p$-HYPERCENTER.

In this section we denote by $\underline{F}_p$ the information of p'-nilpotent groups. As observed in section 2, $\underline{F}_p$ is a saturated subgroup closed formation with an integrated local definition. In general $O_p(G) \leq Z_{\underline{F}_p}(G)$ as S_4 shows with p = 2. In this section we sometimes consider groups from the class

$$\underline{F}_1 = \{ G \; ; \; O_p(G) \leq Z_{\underline{F}_p}(G) \}.$$

It is well known that hypercenter $Z_\infty(G)$ can be characterized as follows:

(i) intersection of all maximal nilpotent subgroups of G,

(ii) intersection of the normalizers of all Sylow subgroups of G.

We obtain two similar characterizations for $Z_{\underline{F}_p}(G)$ when $G \in \underline{F}_1$, G solvable. Using one of these characterizations we obtain a condition for a group to be p'-nilpotent.

THEOREM 5.1 : Let G be solvable, $G \in \underline{F}_1$. Then $Z_{\underline{F}_o}(G) = \bigcap_{q \neq p} \{N_G(S^q) \; ; \; S_q$ is a Sylow q-complement$\}$.

PROOF : Suppose $Z_{\underline{F}_p}(G) = 1$. Since $G \in \underline{F}_1$, $O_p(G) \leq Z_{\underline{F}_p}(G)$. Hence $O_p(G) = 1$. Let $D = \bigcap_{q \neq p} \{ N_G(S^q) \; ; \; S^q$ is a Sylow q-complement$\}$. Suppose $D \neq 1$. Clearly $D \lhd G$ and for $q \neq p$, $D \cap S^q = D^q \lhd D$. Thus D is q-nilpotent $\forall\, q \neq p$, so D is p'-nilpotent. D_p char $D \lhd G$ implies $D_p \lhd G$. Hence $D_p \leq O_p(G) = 1$. Thus D is of p-free order and hence D is nilpotent. Let $N \leq D$, N a minimal normal subgroup of G. N is an r-group with $r \neq p$, and since $N \leq N_G S^r)$ with $(|N| \; , \; |S^r|) = 1$, we see that $[S^r \; , \; N] = 1$. $N \lhd G_r$ implies $N \cap Z(G_r) \neq 1$. Hence there exists $x \neq 1$, $x \in N \cap Z(G_r)$ with $S_r C_G(N) \leq C_G(x)$. i.e., $G = S_r S^r \leq S_r C_G(N) \leq C_G(x)$. Hence $C_G(x) = G$. Thus $N = \langle x \rangle \leq Z(G) \leq Z_{\underline{F}_p}(G) = 1$. This is contrary to $N \neq 1$. Hence $D = 1$. Assume now that $Z_{\underline{F}_p}(G) \neq 1$. Let N be a minimal normal subgroup of G contained in $Z_{\underline{F}_p}(G)$. We now consider two cases.

CASE 1. N is a p-group.

In G/N, by induction on $|G|$, we have $Z_{\underline{F}_p}(G/N) = \bigcap_{q \neq p} \{N_{G/N}(S^q/N)\}$. Since the definition of $Z_{\underline{F}_p}(G)$ is based on the chief factors, we see that $Z_{\underline{F}_p}(G/N) = Z_{\underline{F}_p}(G)/N$. Also, $N_{G/N}(S^q/N) = (N_G(S^q))/N$. Thus $Z_{\underline{F}_p}(G)/N = (\bigcap_{q \neq p} N_G(S^q))/N$; i.e., $Z_{\underline{F}_p}(G) = \bigcap_{q \neq p} (N_G(S^q))$.

CASE 2. N is an r-group, $r \neq p$.

Since $N \leq Z_{\underline{F}_p}(G)$, we have $N \leq Z(G)$ using 4.1. Hence $N \leq N_G(S^q)$ $\forall\, q$. Therefore, $Z_{\underline{F}_p}(G/N) = \bigcap_{q \neq p} (N_{G/N}(S^q/N))$. As in case 1, the result now follows. Q.E.D.

It is easy to verify that if M and N are normal p'-nilpotent subgroups of G, then MN is a normal p'-nilpotent subgroup of G. However, if we drop the normality requirement on one of the subgroups, say M, then MN is still a subgroup, but not necessarily p'-nilpotent. Consider $G = S_4$, $M = G_2$, $N = A_4$. M is 2'-nilpotent, N is 2'-nilpotent normal in G. However G = MN is not 2'-nilpotent. We prove in the next theorem that if M is p'-nilpotent and $N \lhd G$ with $N \leq Z_{F_p}(G)$, then MN is p'-nilpotent.

THEOREM 5.2 : Let M be a p'-nilpotent subgroup of G, $N \lhd G$, $N \leq Z_{F_p}(G)$. Then MN is p'-nilpotent.

PROOF : Let L be a minimal normal subgroup of G contained in N. Consider G/L. By induction on $|G|$, $(ML/L)\cdot(N/L)$ is p'-nilpotent in G/L.

CASE 1. L is a p-group

$(MN)_p/L \simeq MN/L$ since MN/L is p'-nilpotent. $(MN)^pL/L \simeq (MN)^p$ is nilpotent. Thus, $(MN)/(MN)_p$ is nilpotent. and hence MN is p'-nilpotent.

CASE 2. L is a q-group, $q \neq p$.

Using 4.1, $L \leq Z(G)$. By induction on $|G|$, MN/L is p'-nilpotent. $(MN)_pL/L \lhd MN/L$ implies $(MN)_p \lhd MN$ since $L \leq Z(G)$. Also, $(MN/L)^q = (MN/L)^qL/L \lhd MN/L$ $\forall\, q \neq p$. Hence $(MN)^q \lhd MN$ since $L \leq Z(G)$; i.e., MN is q-nilpotent $\forall\, q \neq p$ and hence MN is nilpotent by 2.4. Q.E.D.

We now use this theorem to obtain a description for $Z_{F_p}(G)$ as the intersection of all maximal p'-nilpotent subgroups of G.

THEOREM 5.3 : Let $G \in F_1$. Then $Z_{F_p}(G)$ is the intersection of a maximal p'-nilpotent subgroups of G.

PROOF : Let $C = \bigcap(H$; H is a maximal p'-nilpotent subgroup of G). Suppose $Z_{F_p}(G) = 1$. We now show that C = 1. Clearly $C \lhd G$. Suppose $C \neq 1$. Since $C \leq H$, C is p'-nilpotent. C_p char $C \lhd G$ implies that $C_p \lhd G$. Thus $C_p \leq O_p(G) \leq Z_{F_p}(G) = 1$ implies $C_p = 1$. Therefore, C is nilpotent. Now using an argument similar to that used in the proof of 5.1 we will arrive at a contradiction to the assumption that $C \neq 1$.

(1) There exists a one to one correspondence between the maximal p'-nilpotent subgroups of G and of G/N, N as in 5.1.

For, by 5.2, $N \leq H$ for every maximal p'-nilpotent subgroup H. Suppose K/N is a maximal p'-nilpotent subgroup of G/N. If N is a p-group, then K/N = $(K_p/N)\cdot(K^pN/N)$ where $K_p/N \lhd K/N$ and $K^pN/N \simeq K^p$ is nilpotent. Thus K is a p'-nilpotent subgroup of G, hence a maximal p'-nilpotent subgroup of G. If N is a q-group, $q \neq p$, then $N \leq Z(G)$ by 4.1. Hence K/N p'-nilpotent implies K p'-nilpotent as shown in the proof of 5.2. Thus K is a maximal p'-nilpotent subgroup of G whenever K/N is a maximal p'-nilpotent subgroup of G/N.

(2) Consider G/N and apply induction on $|G|$. Thus, $Z_{F_p}(G/N) = \bigcap(H/N$; H/N is maximal p'-nilpotent in G/N). i.e., $Z_{F_p}(G)/N = \bigcap(H$; H is maximal p'-nilpotent

in G)/N. Hence $Z_{\underline{F}_p}(G) = \bigcap(H$; H is maximal p'-nilpotent in G). Q.E.D.

Next we obtain a condition for a p'-element to lie in $Z_{\underline{F}_p}(G)$.

THEOREM 5.4 : Let G be a p-closed group, $G \in \underline{F}_1$. Let g be a p'-element in G. Then the following are equivalent:

(i) $g \in Z_{\underline{F}_p}(G)$,

(ii) for every p'-element x in G with $(|x|, |g|) = 1$, there exists y in G such that $x^y g = gx^y$.

PROOF : The theorem is trivially true if $Z_{\underline{F}_p}(G) = 1$. So assume that $Z_{\underline{F}_p}(G) \neq 1$. Assume that $g \in Z_{\underline{F}_p}(G)$. $G \in \underline{F}_1$ shows that $O_p(G) \leq Z_{\underline{F}_p}(G)$. Further, $Z_{\underline{F}_p}(G)$ is p'-nilpotent. Moreover, all p'-chief factors of G that are contained in $Z_{\underline{F}_p}(G)$ are central. If $O_p(G) = 1$, then $Z_{\underline{F}_p}(G) = Z_\infty(G)$. Using a well known property of hypercenter, we have $gx = xg$. If $O_p(G) \neq 1$, $O_p(G) = G_p$ since G is p-closed. By definition $G/G_{\underline{F}_p}$ is p'-nilpotent, where $G_{\underline{F}_p}$ is the $\underline{F}_p$-residual of G. Let $gG_{\underline{F}_p}$, $xG_{\underline{F}_p}$ be p'-elements of relative prime orders. By induction on $|G|$,

$(gG_{\underline{F}_p})(x^yG_{\underline{F}_p}) = (x^yG_{\underline{F}_p})(gG_{\underline{F}_p})$ for a suitable $y \in G$; i.e., $[g, x^y] \in G_{\underline{F}_p}$.

Consider $\overline{G} = G/G_p$. By induction on $|G|$, $g^{y_2}G_p$ and $x^{y_1}G_p$ commute for some suitable y_1, y_2 in G such that $y_1 = y\, y_2$. i.e., $[g^{y_2}, x^{y_1}] \in G_p$, i.e., $[g, x^{y_1y_2^{-1}}] \in G_p$, i.e., $[g, x^y] \in G_p$. Using Satz 1.3, p.562 of [8] we note that $gc = cg$ where $c = [g, x^y] \in G_{\underline{F}_p}$ and $g \in Z_{\underline{F}_p}(G)$. $g^{-1}x^y g = [g, x^y]^{-1} = x^y c^{-1}$. Therefore, $\forall\, k \geq 0$, $g^{-k}x^y g^k = x^y c^{-k}$. In particular, for $|g| = m$, $g^{-m}x^y g^m = x^y c^{-m}$, i.e., $c^{-m} = 1$. Since $c \in G_p$ and $(p, m) = 1$, we have $c = 1$, i.e., $g\,x^y = x^y g$. Hence (i) implies (ii).

For proving the reverse implication we consider $\overline{G} = G/Z_{\underline{F}_p}(G)$. Let $\overline{g} = gZ_{\underline{F}_p}(G)$, $\overline{x} = xZ_{\underline{F}_p}(G)$, $|\overline{g}| = m$, $|\overline{x}| = n$, $(m, n) = 1$.

Let $\pi_1 = \{$ distinct primes dividing m $\}$

$\pi_2 = \{$ distinct primes dividing n $\}$

$\langle g\rangle = \langle g_1\rangle \times \langle g_2\rangle$, $\langle x\rangle = \langle x_1\rangle \times \langle x_2\rangle$ where $\langle g_1\rangle = \langle g\rangle_{\pi_1}$, $\langle x_1\rangle = \langle x\rangle_{\pi_2}$. Now applying (ii) for g_1, x_1 we have $g_1x_1^{y_1} = x_1^{y_1}g_1$ for a suitable $y_1 \in G$. By choice of m,n we have $g^m \in Z_{\underline{F}_p}(G)$, $x^n \in Z_{\underline{F}_p}(G)$. Since $\langle x_2\rangle = \langle x\rangle_{\pi_2'}$, $x_2 = x^{m^*}$ where

$m^* = |x_1|$, $x_2 \in Z_{\underline{F}_p}(G)$. Similarly $g_2 \in Z_{\underline{F}_p}(G)$. We noted earlier that $g_1 x_1^{y_1} = x_1^{y_1} g_1$. Hence $(g_1 Z_{\underline{F}_p}(G)(x_1^{y_1} Z_{\underline{F}_p}(G)) = (x_1^{y_1} Z_{\underline{F}_p}(G))(g_1 Z_{\underline{F}_p}(G))$. Since x_2 , $g_2 \in Z_{\underline{F}_p}(G)$, the above equation yields, $(g Z_{\underline{F}_p}(G))(x^{y_1} Z_{\underline{F}_p}(G)) = (x^{y_1} Z_{\underline{F}_p}(G))(g Z_{\underline{F}_p}(G))$. i.e., $\bar{g} = g Z_{\underline{F}_p}(G) \in Z_{\underline{F}_p}(\bar{G}) = \bar{1}$, i.e., $g \in Z_{\underline{F}_p}(G)$. Hence (ii) implies (i). Q.E.D.

We now give an example to show that the condition that G be p-closed in 5.4 is essential.

EXAMPLE 5.5 : Let $A = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle$, $a_i^2 = 1$, $i = 1,2,3$. $B = \langle b ; b^3 = 1 \rangle$, $C = A \times B$. $D = \langle d ; d^7 = 1 \rangle$. $G = [C]D$, $a_1^d = a_1 a_2$, $a_2^d = a_3$, $a_3^d = a_1$, $b^d = b$. $|G| = |C| \cdot |D| = 24 \cdot 7 = 168$. $Z(G) = B$, $G_{2,7} = AD \lhd G$. Consider $G/Z(G)$. This is of order 56. One Sylow 7-subgroup of $G/Z(G)$ is $DZ(G)/Z(G)$. Using Sylow's theorem, the number of Sylow 7-subgroups of $G/Z(G)$ is of the form $1 + 7k$ and $1 + 7k$ divides 8. If $DZ(G)/Z(G) \lhd G/Z(G)$, then $DZ(G) \lhd G$. D char $DZ(G) \lhd G$ implies $D \lhd G$, but $D \ntriangleleft G$. Hence $1 + 7k \neq 1$ and hence $1 + 7k = 8$. i.e., $[G : N_G(G_7)] = 8$. $G_{2,7} = G^3$, Sylow 3-complement in G. $G_{3,7} = G^2$, Sylow 2-complement in G. $[G : N_G(G^2)]$ = number of Sylow 2-complements in G = 8 implies $G^2 = N_G(G^2)$. Let $\underline{G}$ be the formation of 7'-nilpotent groups.

$$\begin{aligned} Z_{\underline{G}}(G) &= \bigcap\{ N_G(G^2) \} \cap \{ N_G(G^3) \} \\ &= \bigcap\{ N_G(G^2) \}, \text{ since } G^3 \lhd G. \\ &= B = Z(G). \end{aligned}$$

Thus $Z_{\underline{G}}(G) = Z_\infty(G) = Z(G) = B$, $O_7(G) = 1 \leqq Z_{\underline{G}}(G)$. Clearly G is not 7-closed. Every 2-element commutes with every 3-element but yet no 2-element lies in $Z_{\underline{G}}(G)$.

THEOREM 5.6 : Let G be a solvable group, $G \in \underline{F}_1$. G is p'-nilpotent if and only if

(i) G is p-closed,

(ii) for every pair of '-elements x,y of relatively prime orders, there exists g in G such that $x\, y^g = y^g\, x$.

PROOF : Assume that G is '-nilpotent. It is a simple matter to verify that (i) and (ii) are satisfied.

Conversely, assume that G satisfies (i) and (ii). Using 5.4, we see that all p'-elements of G lie in $Z_{\underline{F}_p}(G)$. Since $G \in \underline{F}_1$, $O_p(G) \leqq Z_{\underline{F}_p}(G)$. By (i) $O_p(G) = G_p$. Thus $Z_{\underline{F}_p}(G) = G^p G_p = G$. Since $Z_{\underline{F}_p}(G)$ is p'-nilpotent, G is p'-nilpotent. Q.E.D.

REMARK : Example 5.5 shows that we can not drop (i) in the statement of 5.6.

We conclude this paper by obtaining a generating set for the $\underline{F}_p$-residual of G.

THEOREM 5.7 : Let G be a solvable p-closed group with $G \in \underline{F}_1$. Then $G_{\underline{F}_p}$ = $< [x , y^g] ;$ x,y are p'-elements of relatively prime orders and g is a suitable element in G >.

PROOF : Let $N = < [x , y^g] ;$ x,y,g as in statement >. By definition $G/G_{\underline{F}_p}$ is p'-nilpotent. Using 5.6 we have $N \leqq G_{\underline{F}_p}$. Let $\overline{G} = G/N$. Take $\overline{x} = xN$ and $\overline{y} = yN$. Using an argument as in the proof of 5.4 we have $[x , y^g] \in N$.

i.e., $\overline{x}\,\overline{y}^{\overline{g}} = \overline{y}^{\overline{g}}\,\overline{x}$. Now applying 5.6 we see that $\overline{G}$ is p'-nilpotent and so $G_{\underline{F}_p} \leqq N$. Thus $G_{\underline{F}_p} = N$. Q.E.D.

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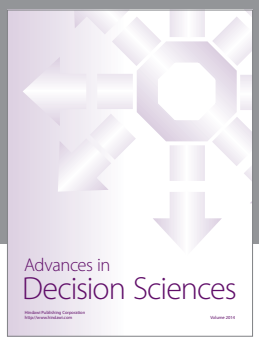

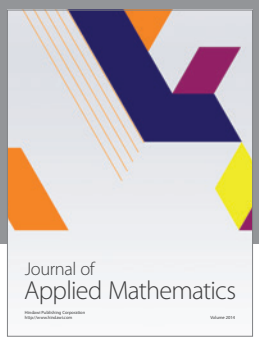

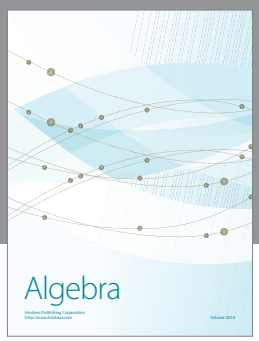

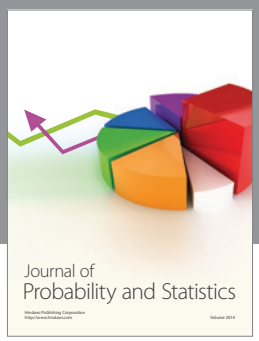

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