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PROPERTIES OF SOME WEAK FORMS OF CONTINUITY

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ABSTRACT. As weak forms of continuity in topological spaces, weak continuity [1], quasi continuity [2], semi continuity [3] and almost continuity in the sense of Husain [4] are well-known. Recently, the following four weak forms of continuity have been introduced: weak quasi continuity [5], faint continuity [6], subweak continuity [7] and almost weak continuity [8]. These four weak forms of continuity are all weaker than weak continuity. In this paper we show that these four forms of continuity are respectively independent and investigate many fundamental properties of these four weak forms of continuity by comparing those of weak continuity, semi continuity and almost continuity.

KEY WORDS AND PHRASES. *weakly continuous, semi continuous, almost continuous, weakly quasi continuous, faintly continuous, subweakly continuous, almost weakly continuous.*
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1. INTRODUCTION.

The notion of continuity is one of the most important tools in Mathematics and many different forms of generalizations of continuity have been introduced and investigated. Weak continuity [1], quasi continuity [2], semi continuity [3] and almost continuity in the sense of Husain [4] are well-known. It is shown in [9] that quasi continuity is equivalent to semi continuity. It will be shown that weak continuity, semi continuity and almost continuity are respectively independent. In 1973, Popa and Stan [5] introduced weak quasi continuity which is implied by both weak continuity and quasi continuity. Recently, faint continuity and subweak continuity which are both implied by weak continuity have been introduced by Long and Herrington [6] and Rose [7], respectively. Quite recently, Janković [8] introduced almost weak continuity as a generalization of both weak continuity and almost continuity. In [10], Piotrowski investigated and compared many properties of quasi continuity, almost continuity and other related weak forms of continuity.

The main purpose of this paper is to show that these four weak forms of continuity implied by weak continuity are respectively independent and to investigate many fundamental properties of such weak forms of continuity by comparing with weak continuity, semi continuity and almost continuity. In Section 3, we obtain some characterizations of almost weak continuity and some relations between almost weak continuity and weak continuity (or almost continuity). Section 4 deals with some characterizations of weakly quasi continuous functions. In Section 5, it is shown that weak quasi continuity, faint continuity, subweak continuity and almost weak continuity are respectively independent. In Section 6, we compare many fundamental properties of semi continuity, almost continuity, weak continuity, subweak continuity, faint continuity, weak quasi continuity and almost weak continuity. The last section is devoted to open questions concerning subweak continuity and faint continuity.

2. PRELIMINARIES.

Throughout this paper spaces always mean topological spaces on which no separation axiom is assumed. By $f : X \to Y$ we denote a function f of a topological space X into a topological space Y. Let S be a subset of a space. The closure and the interior of S are denoted by $Cl(S)$ and $Int(S)$, respectively. A subset S is said to be *semi-open* [3] (resp. *regular closed*, an *α-set* [11]) if $S \subset Cl(Int(S))$ (resp. $S = Cl(Int(S))$, $S \subset Int(Cl(Int(S)))$). The family of all semi-open (resp. regular closed) sets in a space X is denoted by $SO(X)$ (resp. $RC(X)$). The complement of a semi-open set is called *semi-closed*. The intersection of all semi closed sets containing S is called the *semi-closure* of S [12] and is denoted by $sCl(S)$. The union of all semi-open sets contained in S is called the *semi-interior* [12] and is denoted by $sInt(S)$. A subset S is said to be *θ-open* [6] if for each $x \in S$ there exists an open set U such that $x \in U \subset Cl(U) \subset S$.

DEFINITION 2.1. A function $f : X \to Y$ is said to be *semi continuous* [3] (resp. *α-continuous* [13]) if for every open set V of Y, $f^{-1}(V)$ is a semi-open set (resp. an α-set) of X.

A function $f : X \to Y$ is said to be *quasi continuous* at $x \in X$ [2] if for each open set V containing $f(x)$ and each open set U containing x, there exists an open set G of X such that $\emptyset \neq G \subset U$ and $f(G) \subset V$. If f is quasi continuous at every $x \in X$, then it is called *quasi continuous*. In [9, Theorem 1.1], it is shown that a function is semi continuous if and only if it is quasi continuous.

DEFINITION 2.2. A function $f : X \to Y$ is said to be *weakly continuous* [1] if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset Cl(V)$.

DEFINITION 2.3. A function $f : X \to Y$ is said to be *almost continuous* [4] if for each $x \in X$ and each open set V containing $f(x)$, $Cl(f^{-1}(V))$ is a neighborhood of x.

In [13, Theorem 3.2], it is shown that a function is α-continuous if and only if it is almost continuous and semi continuous. In [14] (resp. [10]), almost continuous functions are called *precontinuous* (resp. *nearly continuous*).

DEFINITION 2.4. A function $f : X \to Y$ is said to be *weakly quasi continuous* [5] at $x \in X$ if for each open set V containing $f(x)$ and each open set U containing x, there exists an open set G of X such that $\emptyset \neq G \subset U$ and $f(G) \subset Cl(V)$. If f is weakly quasi continuous at every $x \in X$, then it is called *weakly quasi continuous* (briefly w.q.c.).

Both weak continuity and semi continuity imply weak quasi continuity but the converses are not true by Examples 5.2 and 5.10 (below).

DEFINITION 2.5. A function $f : X \to Y$ is said to be *faintly continuous* (briefly f.c.) [6] if for every θ-open set V of Y, $f^{-1}(V)$ is open in X.

It is shown in [6] that every weakly continuous function is faintly continuous but not conversely.

DEFINITION 2.6. A function $f : X \to Y$ is said to be *subweakly continuous* (briefly s.w.c.) [7] if there exists an open basis Σ for the topology of Y such that $Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for each $V \in \Sigma$.

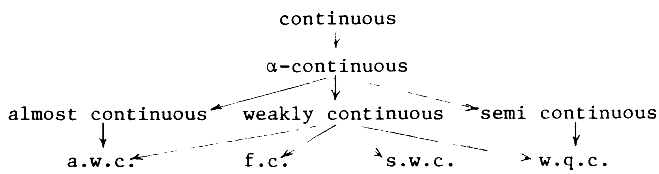
It is shown in [7] that every weakly continuous function is subweakly continuous but not conversely.

DEFINITION 2.7. A function $f : X \to Y$ is said to be *almost weakly continuous* (briefly a.w.c.) [8] if $f^{-1}(V) \subset Int(Cl(f^{-1}(Cl(V))))$ for every open set V of Y.

A function $f : X \to Y$ is weakly continuous if and only if for every open set V of Y, $f^{-1}(V) \subset Int(f^{-1}(Cl(V)))$ [1, Theorem 1]. A function $f : X \to Y$ is almost continuous if and only if $f^{-1}(V) \subset Int(Cl(f^{-1}(V)))$ for every open set V of Y [7, Theorem 4]. Therefore, almost weak continuity is implied by both weak continuity and almost continuity.

From some remarks and definitions previously stated, we obtain the following diagram. In Section 5, it will be shown that the four weak forms of continuity which are all weaker than weak continuity are respectively independent.

DIAGRAM



3. ALMOST WEAKLY CONTINUOUS FUNCTIONS.

In this section, we obtain some characterizations of a.w.c. functions and some relations between almost weak continuity and almost continuity (or weak continuity).

THEOREM 3.1. For a function $f : X \to Y$ the following are equivalent:

(a) f is a.w.c.

(b) $Cl(Int(f^{-1}(V))) \subset f^{-1}(Cl(V))$ for every open set V of Y.

(c) For each $x \in X$ and each open set V containing $f(x)$, $Cl(f^{-1}(Cl(V)))$ is a neighborhood of x.

PROOF. (a) $\to$ (b): Let V be an open set of Y. Then $Y - Cl(V)$ is open in Y and we have

$$X - f^{-1}(Cl(V)) = f^{-1}(Y - Cl(V))$$
$$\subset Int(Cl(f^{-1}(Cl(Y - Cl(V))))) \subset X - Cl(Int(f^{-1}(V))).$$

Therefore, we obtain $Cl(Int(f^{-1}(V))) \subset f^{-1}(Cl(V))$.

(b) $\to$ (c): Let $x \in X$ and V an open set containing $f(x)$. Since $Y - Cl(V)$ is open in Y, we have

$$X - Int(Cl(f^{-1}(Cl(V)))) = Cl(Int(f^{-1}(Y - Cl(V)))) \subset f^{-1}(Cl(Y - Cl(V)))$$
$$= f^{-1}(Y - Int(Cl(V))) \subset f^{-1}(Y - V) = X - f^{-1}(V).$$

Therefore, we obtain $x \in f^{-1}(V) \subset Int(Cl(f^{-1}(Cl(V))))$ and hence $Cl(f^{-1}(Cl(V)))$ is a neighborhood of x.

(c) $\to$ (a): Let V be any open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$ and $Cl(f^{-1}(Cl(V)))$ is a neighborhood of x. Therefore, $x \in Int(Cl(f^{-1}(Cl(V))))$ and we obtain $f^{-1}(V) \subset Int(Cl(f^{-1}(Cl(V))))$.

Janković [8] remarked that a.w.c. functions into regular spaces are almost continuous. It will be shown in Example 5.8 (below) that an almost continuous function into a discrete space is not necessarily weakly continuous. Therefore, it is not true in general that if Y is a regular space and $f : X \to Y$ is a.w.c. then f is weakly continuous.

Rose [7] defined a function $f : X \to Y$ to be *almost open* if for every open set U of X, $f(U) \subset \mathrm{Int}(\mathrm{Cl}(f(U)))$ and showed that a function $f : X \to Y$ is almost open if and only if $f^{-1}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(f^{-1}(V))$ for every open set V of Y.

THEOREM 3.2. If a function $f : X \to Y$ is a.w.c. and almost open, then it is almost continuous.

PROOF. Let $x \in X$ and V an open set containing $f(x)$. By Theorem 11 of [7] we have $x \in f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V)))) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$. Therefore, $\mathrm{Cl}(f^{-1}(V))$ is a neighborhood of x and hence f is almost continuous.

COROLLARY 3.3 (Rose [7]). Every weakly continuous and almost open function is almost continuous.

An a.w.c. and almost open function is not necessarily weakly continuous since the function in Example 5.8 (below) is almost continuous and almost open but not weakly continuous. It will be shown in Examples 5.2 and 5.8 that semi continuity and almost weak continuity are independent of each other. Therefore, semi continuity does not imply weak continuity. However, we have

THEOREM 3.4. If a function $f : X \to Y$ is a.w.c. and semi continuous, then it is weakly continuous.

PROOF. Let V be an open set of Y. Since f is semi continuous, we have $f^{-1}(V) \in SO(X)$ and hence $\mathrm{Cl}(f^{-1}(V)) = \mathrm{Cl}(\mathrm{Int}(f^{-1}(V)))$ [15, Lemma 2]. On the other hand, since f is a.w.c., by Theorem 3.1 we have $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subset f^{-1}(\mathrm{Cl}(V))$ and hence $\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$. It follows from Theorem 7 of [7] that f is weakly continuous.

4. WEAKLY QUASI CONTINUOUS FUNCTIONS.

In this section, we obtain some characterizations of w.q.c. functions.

THEOREM 4.1. A function $f : X \to Y$ is w.q.c. if and only if for each $x \in X$ and each open set V containing $f(x)$, there exists $U \in SO(X)$ containing x such that $f(U) \subset \mathrm{Cl}(V)$.

PROOF. Necessity. Suppose that f is w.q.c. Let $x \in X$ and V an open set containing $f(x)$. Let Λ be the family of all open neighborhoods of x in X. Then for each $N \in \Lambda$ there exists an open set G_N of X such that $\emptyset \neq G_N \subset N$ and $f(G_N) \subset \mathrm{Cl}(V)$. Put $G = \bigcup\{G_N \mid N \in \Lambda\}$, then G is open in X and $x \in \mathrm{Cl}(G)$. Let $U = G \bigcup \{x\}$, then we have $x \in U \in SO(X)$ and $f(U) \subset \mathrm{Cl}(V)$.

Sufficiency. Let $x \in X$, U be an open set containing x and V an open set containing $f(x)$. There exists an $A \in SO(X)$ containing x such that $f(A) \subset \mathrm{Cl}(V)$. Put $G = \mathrm{Int}(A \cap U)$. Then, by Lemmas 1 and 4 of [15], G is a nonempty open set of X such that $G \subset U$ and $f(G) \subset \mathrm{Cl}(V)$. This shows that f is w.q.c.

THEOREM 4.2. A function $f : X \to Y$ is w.q.c. if and only if for every $F \in RC(Y)$ $f^{-1}(F) \in SO(X)$.

PROOF. Necessity. Suppose that f is w.q.c. Let $F \in RC(Y)$. By Theorem 2 of [5], we have $f^{-1}(F) = f^{-1}(\mathrm{Cl}(\mathrm{Int}(F))) \subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(\mathrm{Cl}(\mathrm{Int}(F))))) \subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(F)))$. Therefore, we obtain $f^{-1}(F) \in SO(X)$.

Sufficiency. Let V be an open set of Y. Since $\mathrm{Cl}(V) \in RC(Y)$, we have

$f^{-1}(Cl(V)) \in SO(X)$ and hence $f^{-1}(Cl(V)) \subset Cl(Int(f^{-1}(Cl(V))))$. It follows from Theorem 2 of [5] that f is w.q.c.

THEOREM 4.3. For a function $f : X \to Y$ the following are equivalent:

(a) f is w.q.c.

(b) $sCl(f^{-1}(Int(Cl(B)))) \subset f^{-1}(Cl(B))$ for every subset B of Y.

(c) $sCl(f^{-1}(Int(F))) \subset f^{-1}(F)$ for every $F \in RC(Y)$.

(d) $sCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every open set V of Y.

(e) $f^{-1}(V) \subset sInt(f^{-1}(Cl(V)))$ for every open set V of Y.

PROOF. (a) $\to$ (b): Let B be a subset of Y. Assume that $x \notin f^{-1}(Cl(B))$. Then $f(x) \notin Cl(B)$ and there exists an open set V containing $f(x)$ such that $V \cap B = \emptyset$; hence $Cl(V) \cap Int(Cl(B)) = \emptyset$. By Theorem 4.1, there exists $U \in SO(X)$ containing x such that $f(U) \subset Cl(V)$. Therefore, we have $U \cap f^{-1}(Int(Cl(B))) = \emptyset$ and hence $x \notin sCl(f^{-1}(Int(Cl(B))))$. Thus, we obtain

$$sCl(f^{-1}(Int(Cl(B)))) \subset f^{-1}(Cl(B)).$$

(b) $\to$ (c): Let $F \in RC(Y)$. By (b), we have

$$\begin{aligned} sCl(f^{-1}(Int(F))) &= sCl(f^{-1}(Int(Cl(Int(F))))) \\ &\subset f^{-1}(Cl(Int(F))) = f^{-1}(F). \end{aligned}$$

(c) $\to$ (d): For an open set V of Y, $Cl(V) \in RC(Y)$ and by (c) we have

$$sCl(f^{-1}(V)) \subset sCl(f^{-1}(Int(Cl(V)))) \subset f^{-1}(Cl(V)).$$

(d) $\to$ (e): Let V be an open set of Y and $x \notin sInt(f^{-1}(Cl(V)))$. Then

$$x \in X - sInt(f^{-1}(Cl(V))) = sCl(f^{-1}(Y - Cl(V))).$$

Since $Y - Cl(V)$ is open in Y, by (d) we have

$$\begin{aligned} sCl(f^{-1}(Y - Cl(V))) &\subset f^{-1}(Cl(Y - Cl(V))) \\ &= f^{-1}(Y - Int(Cl(V))) \subset X - f^{-1}(V). \end{aligned}$$

Therefore, we obtain $x \notin f^{-1}(V)$ and hence $f^{-1}(V) \subset sInt(f^{-1}(Cl(V)))$.

(e) $\to$ (a): Let $x \in X$ and V be an open set containing $f(x)$. We have

$$x \in f^{-1}(V) \subset sInt(f^{-1}(Cl(V))) \in SO(X).$$

Put $U = sInt(f^{-1}(Cl(V)))$. Then, we obtain $x \in U \in SO(X)$ and $f(U) \subset Cl(V)$. It follows from Theorem 4.1 that f is w.q.c.

5. EXAMPLES.

In this section, we shall show that semi continuity, almost continuity and weak continuity are respectively independent. Moreover, it will be shown that each two of quasi weak continuity, faint continuity, almost weak continuity and subweak continuity are independent of each other. It is shown in Theorem 2 of [1] that if $f : X \to Y$ is weakly continuous and Y is regular then f is continuous. Theorem 11 of [6] shows that "weakly continuous" in the above result can be replaced by "f.c.". However, we shall observe that "weakly continuous" in the above result can not be replaced by "semi continuous", "almost continuous", "s.w.c.", "w.q.c." or "a.w.c.".

REMARK 5.1. There exists a semi continuous function into a regular space which is neither f.c., s.w.c. nor a.w.c. Therefore, semi continuity implies neither weak continuity nor almost continuity.

EXAMPLE 5.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then (X, σ) is a regular space. Since $\{b, c\} \in SO(X, \tau)$, f is semi continuous and hence w.q.c. However, f is neither f.c., s.w.c. nor a.w.c.

REMARK 5.3. There exists a f.c. function which is neither w.q.c., s.w.c. nor a.w.c. The following example is due to Long and Herrington [6].

EXAMPLE 5.4. Let $X = \{0, 1\}$ and $\tau = \{\emptyset, X, \{1\}\}$. Let $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(0) = a$ and $f(1) = b$. Then f is f.c. [6, Example 2]. However, f is neither w.q.c., s.w.c. nor a.w.c.

REMARK 5.5. There exists a s.w.c. function into a discrete space which is neither w.q.c., f.c. nor a.w.c. Therefore, a s.w.c. function is not necessarily weakly continuous even if the range is a regular space.

EXAMPLE 5.6. Let X be the set of all real numbers, τ the countable complement topology for X and σ the discrete topology for X. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is s.w.c. since the set $\{\{x\} \mid x \in X\}$ is an open basis for σ and (X, τ) is T_1. However, f is neither w.q.c., f.c. nor a.w.c.

REMARK 5.7. There exists an almost continuous function into a regular space which is neither w.q.c., f.c. nor s.w.c. Therefore, almost continuity implies neither weak continuity nor semi continuity.

EXAMPLE 5.8. Let X be the real numbers with the indiscrete topology, Y the real numbers with the discrete topology and $f : X \to Y$ the identity function. Then f is almost continuous and hence a.w.c. However, f is neither w.q.c., f.c. nor s.w.c.

REMARK 5.9. There exists a weakly continuous function which is neither semi continuous nor almost continuous.

EXAMPLE 5.10. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}, \{b, c, d\}\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ as follows: $f(a) = c$, $f(b) = d$, $f(c) = b$ and $f(d) = a$. Then f is weakly continuous [16, Example]. However, f is neither semi continuous nor almost continuous since there exists $\{c\} \in \tau$ such that $f^{-1}(\{c\}) = \{a\}$ and $\mathrm{Int}(\{a\}) = \mathrm{Int}(\mathrm{Cl}(\{a\})) = \emptyset$.

6. PROPERTIES OF SEVEN WEAK FORMS OF CONTINUITY.

In this section, we investigate the behavior of seven weak forms of continuity under the operations like compositions, restrictions, graph functions, and generalized products. And also we study if connectedness and hyperconnectedness are preserved under such functions. Many results stated below concerning semi continuity, weak continuity and almost continuity have been already known. Many properties of faint continuity and subweak continuity are also known in [6], [17] and [18]. The known results will be denoted only by numbers with the bracket ( ). In contrast to this, new results will be denoted by THEOREM, LEMMA, EXAMPLE etc.

6.1. COMPOSITIONS.

The following are shown in [3, Example 11] and [18, Example 2].

(6.1.1) The composition of two semi continuous (resp. weakly continuous, s.w.c.) functions is not necessarily semi continuous (resp. weakly continuous, s.w.c.).

THEOREM 6.1.2. The composition of two almost continuous functions is not necessarily almost continuous.

PROOF. See the proof of Theorem 6.1.8 (below).

THEOREM 6.1.3. The composition of two w.q.c. (resp. a.w.c.) functions is not necessarily w.q.c. (resp. a.w.c.).

PROOF. In Example 2 of [18], f and g are weakly continuous. However, the composition $g \circ f$ is neither w.q.c. nor a.w.c.

In the sequel we investigate the behaviour of compositions in case one of two functions is continuous.

THEOREM 6.1.4. If $f : X \to Y$ is semi continuous (resp. almost continuous) and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is semi continuous (resp. almost continuous).

PROOF. The proof is obvious and is thus omitted.

The next results follow from the facts stated in [18, p. 810 and Lemma 1].

(6.1.5) If $f : X \to Y$ is weakly continuous (resp. s.w.c., f.c.) and $g : Y \to Z$ is continuous, then $g \circ f$ is weakly continuous (resp. s.w.c., f.c.).

THEOREM 6.1.6. If $f : X \to Y$ is w.q.c. (resp. a.w.c.) and $g : Y \to Z$ is continuous, then $g \circ f$ is w.q.c. (resp. a.w.c.).

PROOF. First, by using Theorem 4.1 we show that $g \circ f$ is w.q.c. Let $x \in X$ and W an open set containing $g(f(x))$. Then $g^{-1}(W)$ is an open set containing $f(x)$ and there exists $U \in SO(X)$ containing x such that $f(U) \subset Cl(g^{-1}(W))$. Since g is continuous, we obtain $(g \circ f)(U) \subset g(Cl(g^{-1}(W))) \subset Cl(W)$. Next, we show that $g \circ f$ is a.w.c. Let W be an open set of Z. Then $g^{-1}(W)$ is open in Y and hence we have $(g \circ f)^{-1}(W) \subset Int(Cl(f^{-1}(Cl(g^{-1}(W))))) \subset Int(Cl((g \circ f)^{-1}(Cl(W))))$. This shows that $g \circ f$ is a.w.c.

THEOREM 6.1.7. The composition $g \circ f$ of a continuous function $f : X \to Y$ and a semi continuous function $g : Y \to Z$ is not necessarily w.q.c.

PROOF. Let $X = Y = Z = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}\}$, $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$ and $\theta = \{\emptyset, Z, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \theta)$ be the identity functions. Then f is continuous and g is semi continuous since $g^{-1}(\{b, c, d\}) \in SO(Y, \sigma)$. The set $\{b, c, d\}$ is regular closed in (Z, θ) and $(g \circ f)^{-1}(\{b, c, d\}) \notin SO(X, \tau)$. Thus, by Theorem 4.2 $g \circ f$ is not w.q.c. and hence not semi continuous.

THEOREM 6.1.8. The composition $g \circ f$ of a continuous function $f : X \to Y$ and an almost continuous function $g : Y \to Z$ is not necessarily a.w.c.

PROOF. Let $X = Y = Z$ be the set of real numbers. Let τ be the usual topology, σ the indiscrete topology and θ the discrete topology. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \theta)$ be the identity functions. Then f is continuous and g is almost continuous by Example 5.8. However, $g \circ f$ is not a.w.c. since $Int(Cl((g \circ f)^{-1}(Cl(\{z\})))) = \emptyset$ for every $\{z\} \in \theta$. Hence $g \circ f$ is not almost continuous.

The following is shown in Lemma 1 of [18].

(6.1.9) If $f : X \to Y$ is continuous and $g : Y \to Z$ is weakly continuous, then $g \circ f$ is weakly continuous.

THEOREM 6.1.10. If $f : X \to Y$ is continuous and $g : Y \to Z$ is s.w.c. (resp. f.c.), then $g \circ f : X \to Z$ is s.w.c. (resp. f.c.).

PROOF. Suppose that f is continuous and g is s.w.c. There exists an open basis Σ of Z such that $Cl(g^{-1}(W)) \subset g^{-1}(Cl(W))$ for every $W \in \Sigma$. Since f is continuous, we have $Cl((g \circ f)^{-1}(W)) \subset f^{-1}(Cl(g^{-1}(W))) \subset (g \circ f)^{-1}(Cl(W))$. Therefore, $g \circ f$ is s.w.c. Suppose that f is continuous and g is f.c. For every θ-open set W of Z, $g^{-1}(W)$ is open in Y and hence $(g \circ f)^{-1}(W)$ is open in X. Hence $g \circ f$ is f.c.

6.2. RESTRICTIONS.

THEOREM 6.2.1. The restriction of a semi continuous function to a regular closed subset is not necessarily w.q.c. and hence it need not be semi continuous.

PROOF. In Example 5.2, $f : (X, \tau) \to (X, \sigma)$ is semi continuous and $A = \{a, c\} \in$

$RC(X, \tau)$. The restriction $f|A : A \to (X, \sigma)$ is not w.q.c. and hence it is not semi continuous.

The following is shown in Example 3 of [19].

(6.2.2) The restriction of an almost continuous function to any subset is not necessarily almost continuous.

THEOREM 6.2.3. If $f : X \to Y$ is weakly continuous and A is a subset of X, then the restriction $f|A : A \to Y$ is weakly continuous.

PROOF. Let V be an open set of Y. Since f is weakly continuous, by Theorem 4 of [20] we have $Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$. Therefore, we obtain

$$Cl_A((f|A)^{-1}(V)) = Cl_A(f^{-1}(V) \cap A) \subset Cl(f^{-1}(V)) \cap A \subset (f|A)^{-1}(Cl(V)),$$

where $Cl_A(B)$ denotes the closure of B in the subspace A. It follows from [7, Theorem 7] that $f|A$ is weakly continuous.

The following are shown in [17, Theorem 4] and [6, Theorem 12].

(6.2.4) The restriction of a s.w.c. (resp. f.c.) function to a subset is s.w.c. (resp. f.c.).

THEOREM 6.2.5. The restriction of an a.w.c. function to a subset is not necessarily a.w.c.

PROOF. In Example 3 of [19], $f : R \to R$ is almost continuous and hence a.w.c. However, the restriction $f|M : M \to R$ is not a.w.c. at $x = 0$.

In the sequel we investigate the case of restrictions to open sets. The following are shown in [15, Theorem 3] and [19, Theorem 4].

(6.2.6) The restriction of a semi continuous (resp. almost continuous) function to an open set is semi continuous (resp. almost continuous).

The following are immediate consequences of Theorem 6.2.3 and (6.2.4).

(6.2.7) The restriction of a weakly continuous (resp. s.w.c., f.c.) function to an open set is weakly continuous (resp. s.w.c., f.c.).

THEOREM 6.2.8. If $f : X \to Y$ is w.q.c. and A is open in X, then the restriction $f|A : A \to Y$ is w.q.c.

PROOF. Let $x \in A$ and V be an open set of Y containing $f(x)$. Since f is w.q.c., by Theorem 4.1 there exists $U \in SO(X)$ containing x such that $f(U) \subset Cl(V)$. Since A is open in X, by Lemma 1 of [15] $x \in A \cap U \in SO(A)$ and $(f|A)(A \cap U) = f(A \cap U) \subset f(U) \subset Cl(V)$. It follows from Theorem 4.1 that $f|A$ is w.q.c.

THEOREM 6.2.9. If $f : X \to Y$ is a.w.c. and A is open in X, then the restriction $f|A : A \to Y$ is a.w.c.

PROOF. Let V be an open set of Y. Since f is a.w.c., we have $f^{-1}(V) \subset Int(Cl(f^{-1}(Cl(V))))$. Since A is open, we obtain

$$(f|A)^{-1}(V) \subset A \cap Int(Cl(f^{-1}(Cl(V)))) = Int_A(A \cap Cl(f^{-1}(Cl(V))))$$
$$\subset Int_A(A \cap Cl(A \cap f^{-1}(Cl(V)))) = Int_A(Cl_A((f|A)^{-1}(Cl(V)))),$$

where $Int_A(B)$ and $Cl_A(B)$ denote the interior and the closure of B in the subspace A, respectively. This shows that $f|A$ is a.w.c.

6.3. GRAPH FUNCTIONS.

Let $f : X \to Y$ be a function. A function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for every $x \in X$, is called the *graph function* of f. The following are shown in [21, Theorem 2], [22, Theorem 2] and [20, Theorem 1].

(6.3.1) The graph function g of a function f is semi continuous (resp. almost continuous, weakly continuous) if and only if f is semi continuous (resp. almost continuous, weakly continuous).

The following is shown in Theorem 7 of [17].

(6.3.2) If a function is s.w.c., then the graph function is s.w.c.

The following is shown in Theorem 13 of [6].

(6.3.3) A function is f.c. if the graph function is f.c.

THEOREM 6.3.4. The graph function $g : X \to X \times Y$ is w.q.c. if and only if $f : X \to Y$ is w.q.c.

PROOF. Necessity. Suppose that g is w.q.c. Let $x \in X$ and V an open set containing $f(x)$. Then $X \times V$ is an open set containing $g(x)$ and by Theorem 4.1 there exists $U \in SO(X)$ containing x such that $g(U) \subset Cl(X \times V)$. Therefore, we obtain $f(U) \subset Cl(V)$ and hence f is w.q.c. by Theorem 4.1.

Sufficiency. Suppose that f is w.q.c. Let $x \in X$ and W be an open set containing $g(x)$. There exist open sets $U_1 \subset X$ and $V \subset Y$ such that $g(x) = (x, f(x)) \in U_1 \times V \subset W$. Since f is w.q.c., by Theorem 4.1 there exists $U_2 \in SO(X)$ containing x such that $f(U_2) \subset Cl(V)$. Put $U = U_1 \cap U_2$, then $x \in U \in SO(X)$ [15, Lemma 1] and $g(U) \subset Cl(W)$. It follows from Theorem 4.1 that g is w.q.c.

THEOREM 6.3.5. The graph function $g : X \to X \times Y$ is a.w.c. if and only if $f : X \to Y$ is a.w.c.

PROOF. Necessity. Suppose that g is a.w.c. In general, we have $g^{-1}(X \times B) = f^{-1}(B)$ for every subset B of Y. Let V be an open set of Y. By Theorem 3.1, we obtain $Cl(Int(f^{-1}(V))) = Cl(Int(g^{-1}(X \times V))) \subset g^{-1}(Cl(X \times V)) = f^{-1}(Cl(V))$. It follows from Theorem 3.1 that f is a.w.c.

Sufficiency. Suppose that f is a.w.c. Let $x \in X$ and W be an open set of $X \times Y$ containing $g(x)$. There exists a basic open set $U \times V$ such that $g(x) \in U \times V \subset W$. Since f is a.w.c., by Theorem 3.1 $Cl(f^{-1}(Cl(V)))$ is a neighborhood of x and $U \cap Cl(f^{-1}(Cl(V))) \subset Cl(U \cap f^{-1}(Cl(V)))$. On the other hand, we have $U \cap f^{-1}(Cl(V)) \subset g^{-1}(U \times Cl(V)) \subset g^{-1}(Cl(W))$. Therefore, $Cl(g^{-1}(Cl(W)))$ is a neighborhood of x and hence g is a.w.c. by Theorem 3.1.

6.4. PRODUCT FUNCTIONS.

Let $\{X_\alpha \mid \alpha \in \nabla\}$ and $\{Y_\alpha \mid \alpha \in \nabla\}$ be any two families of topological spaces with the same index set ∇. The product space of $\{X_\alpha \mid \alpha \in \nabla\}$ (resp. $\{Y_\alpha \mid \alpha \in \nabla\}$) is simply denoted by ΠX_α (resp. ΠY_α). Let $f_\alpha : X_\alpha \to Y_\alpha$ be a function for each $\alpha \in \nabla$. Let $f : \Pi X_\alpha \to \Pi Y_\alpha$ be the product function defined as follows: $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ for every $\{x_\alpha\} \in \Pi X_\alpha$. The natural projection of ΠX_α (resp. ΠY_α) onto X_β (resp. Y_β) is denoted by $p_\beta : \Pi X_\alpha \to X_\beta$ (resp. $q_\beta : \Pi Y_\alpha \to Y_\beta$). The following are shown in [15, Theorem 5], [14, Theorem 2.6] and [18, Theorem 1].

(6.4.1) The function $f : \Pi X_\alpha \to \Pi Y_\alpha$ is semi continuous (resp. almost continuous, weakly continuous) if and only if $f_\alpha : X_\alpha \to Y_\alpha$ is semi continuous (resp. almost continuous, weakly continuous) for each $\alpha \in \nabla$.

The following two results are shown in Theorems 3 and 5 of [18].

(6.4.2) If $f_\alpha : X_\alpha \to Y_\alpha$ is s.w.c. for each $\alpha \in \nabla$, then $f : \Pi X_\alpha \to \Pi Y_\alpha$ is s.w.c.

(6.4.3) If $f : \Pi X_\alpha \to \Pi Y_\alpha$ is f.c., then $f_\alpha : X_\alpha \to Y_\alpha$ is f.c. for each $\alpha \in \nabla$.

LEMMA 6.4.4. Let $f : X \to Y$ be an open continuous surjection and $g : Y \to Z$ a function. If $g \circ f : X \to Z$ is w.q.c., then g is w.q.c.

PROOF. Let $F \in RC(Z)$. Since $g \circ f$ is w.q.c., $(g \circ f)^{-1}(F) \in SO(X)$ by Theorem 4.2. Since f is an open sontinuous surjection, by Theorem 9 of [3] we obtain $f((g \circ f)^{-1}(F)) = g^{-1}(F) \in SO(Y)$. It follows from Theorem 4.2 that g is w.q.c.

THEOREM 6.4.5. The function $f : \Pi X_\alpha \to \Pi Y_\alpha$ is w.q.c. if and only if $f_\alpha : X_\alpha \to Y_\alpha$ is w.q.c. for each $\alpha \in \nabla$.

PROOF. Necessity. Suppose that f is w.q.c. Let $\beta \in \nabla$. Since $q_\beta : \Pi Y_\alpha \to Y_\beta$ is continuous, by Theorem 6.1.6 $f_\beta \circ p_\beta = q_\beta \circ f$ is w.q.c. Moreover, p_β is an open continuous surjection and by Lemma 6.4.4 f_β is w.q.c.

Sufficiency. Let $x = \{x_\alpha\} \in \Pi X_\alpha$ and W be an open set containing $f(x)$. There exists a basic open set ΠV_α such that $f(x) \in \Pi V_\alpha \subset W$, where for a finite number of ∇, say, $\alpha_1, \alpha_2, \dots, \alpha_n$, V_{α_j} is open in Y_{α_j} and otherwise $V_\alpha = Y_\alpha$. Since f_α is w.q.c., there exists $U_\alpha \in SO(X_\alpha)$ containing x_α such that $f_\alpha(U_\alpha) \subset Cl(V_\alpha)$ for $\alpha = \alpha_1, \alpha_2, \dots, \alpha_n$. Put

$$U = \prod_{j=1}^{n} U_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} X_\alpha,$$

then $x \in U \in SO(\Pi X_\alpha)$ [15, Theorem 2] and

$$f(U) \subset \prod_{j=1}^{n} f_{\alpha_j}(U_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} Y_\alpha \subset \prod_{j=1}^{n} Cl(V_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} Y_\alpha \subset Cl(W).$$

Therefore, it follows from Theorem 4.1 that f is w.q.c.

LEMMA 6.4.6. Let $f : X \to Y$ be an open continuous surjection and $g : Y \to Z$ a function. If $g \circ f : X \to Z$ is a.w.c., then g is a.w.c.

PROOF. Let W be an open set of Z. Since $g \circ f$ is a.w.c., we have

$$(g \circ f)^{-1}(W) \subset Int(Cl((g \circ f)^{-1}(Cl(W)))) \subset Int(f^{-1}(Cl(g^{-1}(Cl(W))))).$$

Since f is an open surjection, we obtain $g^{-1}(W) \subset Int(Cl(g^{-1}(Cl(W))))$. This shows that g is a.w.c.

THEOREM 6.4.7. The function $f : \Pi X_\alpha \to \Pi Y_\alpha$ is a.w.c. if and only if $f_\alpha : X_\alpha \to Y_\alpha$ is a.w.c. for each $\alpha \in \nabla$.

PROOF. Necessity. Suppose that f is a.w.c. Let $\beta \in \nabla$. Since f is a.w.c. and $q_\beta : \Pi Y_\alpha \to Y_\beta$ is continuous, by Theorem 6.1.6 $f_\beta \circ p_\beta = q_\beta \circ f$ is a.w.c. and hence f_β is a.w.c. by Lemma 6.4.6.

Sufficiency. Let $x = \{x_\alpha\} \in \Pi X_\alpha$ and W be an open set containing $f(x)$. There exists a basic open set ΠV_α such that

$$f(x) \in \Pi V_\alpha \subset W \text{ and } \Pi V_\alpha = \prod_{j=1}^{n} V_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} Y_\alpha,$$

where V_{α_j} is open in Y_{α_j} for $j = 1, 2, \dots, n$. Since f_α is a.w.c., by Theorem 3.1 $Cl(f_{\alpha_j}^{-1}(Cl(V_{\alpha_j})))$ is a neighborhood of x_{α_j} and

$$\prod_{j=1}^{n} Cl(f_{\alpha_j}^{-1}(Cl(V_{\alpha_j}))) \times \prod_{\alpha \neq \alpha_j} X_\alpha \subset Cl(f^{-1}(Cl(W))).$$

Therefore, $Cl(f^{-1}(Cl(W)))$ is a neighborhood of x and f is a.w.c. by Theorem 3.1.

It is well-known that a function $f : X \to \Pi Y_\alpha$ is continuous if and only if $q_\beta \circ f : X \to Y_\beta$ is continuous for each $\beta \in \nabla$. We investigate if weak forms of continuity have this property.

The following are shown in [15, Theorem 6] and [3, Example 10].

(6.4.8) If a function $f : X \to \Pi Y_\alpha$ is semi continuous, then $q_\beta \circ f : X \to Y_\beta$ is semi continuous for each $\beta \in \nabla$. However, the converse is not true.

THEOREM 6.4.9. A function $f : X \to \Pi Y_\alpha$ is almost continuous if and only if $q_\beta \circ f : X \to Y_\beta$ is almost continuous for each $\beta \in \nabla$.

PROOF. Necessity. Since q_β is continuous, this is an immediate consequence of Theorem 6.1.4.

Sufficiency. Let $x \in X$ and W an open set containing $f(x)$ in ΠY_α. There exists a basic open set ΠV_α such that $f(x) \in \Pi V_\alpha \subset W$, where V_{α_j} is open in Y_{α_j} for $j = 1, 2, \ldots, n$ and otherwise $V_\alpha = Y_\alpha$. Since $q_\beta(f(x)) \in V_\beta$ and $q_\beta \circ f$ is almost continuous for each $\beta \in \nabla$, $\mathrm{Cl}((q_{\alpha_j} \circ f)^{-1}(V_{\alpha_j}))$ is a neighborhood of x for for $j = 1, 2, \ldots, n$ and $\bigcap_{j=1}^{n} \mathrm{Cl}((q_{\alpha_j} \circ f)^{-1}(V_{\alpha_j}))$ is a neighborhood of x. Moreover, we have

$$\bigcap_{j=1}^{n} \mathrm{Cl}((q_{\alpha_j} \circ f)^{-1}(V_{\alpha_j})) \subset \mathrm{Cl}(f^{-1}(\Pi V_\alpha)) \subset \mathrm{Cl}(f^{-1}(W)).$$

Assume that $z \notin \mathrm{Cl}(f^{-1}(\Pi V_\alpha))$. There exists an open set U containing z such that $U \cap f^{-1}(\Pi V_\alpha) = \emptyset$. Therefore, $U \cap (q_{\alpha_k} \circ f)^{-1}(V_{\alpha_k}) = \emptyset$ for some k $(1 \leqq k \leqq n)$. This shows that $z \notin \mathrm{Cl}((q_{\alpha_k} \circ f)^{-1}(V_{\alpha_k}))$ and hence we obtain $z \notin \bigcap_{j=1}^{n} \mathrm{Cl}((q_{\alpha_j} \circ f)^{-1}(V_{\alpha_j}))$. Consequently, $\mathrm{Cl}(f^{-1}(W))$ is a neighborhood of x and hence f is almost continuous.

The following three results are shown in Theorems 2, 4 and 6 of [18].

(6.4.10) A function $f : X \to \Pi Y_\alpha$ is weakly continuous if and only if $q_\beta \circ f : X \to Y_\beta$ is weakly continuous for each $\beta \in \nabla$.

(6.4.11) A function $f : X \to \Pi Y_\alpha$ is s.w.c. if $q_\beta \circ f : X \to Y_\beta$ is s.w.c. for each $\beta \in \nabla$.

(6.4.12) If a function $f : X \to \Pi Y_\alpha$ is f.c., then $q_\beta \circ f : X \to Y_\beta$ is f.c. for each $\beta \in \nabla$.

THEOREM 6.4.13. If a function $f : X \to \Pi Y_\alpha$ is s.w.c., then $q_\beta \circ f : X \to Y_\beta$ is s.w.c. for each $\beta \in \nabla$.

PROOF. Since q_β is continuous, this follows immediately from (6.1.5).

THEOREM 6.4.14. If a function $f : X \to \Pi Y_\alpha$ is w.q.c., then $q_\beta \circ f : X \to Y_\beta$ is w.q.c. for each $\beta \in \nabla$. However, the converse is not true in general.

PROOF. Since q_β is continuous, by Theorem 6.1.6 $q_\beta \circ f$ is w.q.c. In Example 10 of [3], $f_i : X \to X_i$ is semi continuous for $i = 1, 2$. However, a function $f : X \to X_1 \times X_2$, defined as follows: $f(x) = (f_1(x), f_2(x))$ for every $x \in X$, is not w.q.c.

THEOREM 6.4.15. A function $f : X \to \Pi Y_\alpha$ is a.w.c. if and only if $q_\beta \circ f : X \to Y_\beta$ is a.w.c. for each $\beta \in \nabla$.

PROOF. The necessity follows from Theorem 6.1.6. By using Theorem 3.1, we can prove the sufficiency similarly to the proof of Sufficiency of Theorem 6.4.9.

6.5. CLOSED GRAPHS.

For a function $f : X \to Y$, the subset $\{(x, f(x)) \mid x \in X\}$ of the product space $X \times Y$ is called the *graph* of f and is denoted by $G(f)$. It is well known that if $f : X \to Y$ is continuous and Y is Hausdorff then $G(f)$ is closed in $X \times Y$. We shall investigate the behaviour of $G(f)$ in case the assumption "*continuous*" on f is replaced by one of seven weak forms of continuity.

THEOREM 6.5.1. If $f : X \to Y$ is semi continuous and Y is Hausdorff, then $G(f)$ is semi-closed in $X \times Y$ but it is not necessarily closed.

PROOF. By Theorem 3 of [21], $G(f)$ is semi-closed in $X \times Y$. In Example 8 of [3], $f : X \to X^*$ is semi continuous and X^* is Hausdorff. However, $G(f)$ is not closed in $X \times X^*$ because $(1/2, 0) \in \mathrm{Cl}(G(f)) - G(f)$.

COROLLARY 6.5.2. A w.q.c. function into a Hausdorff space need not have a closed graph.

THEOREM 6.5.3. An almost continuous function into a Hausdorff space need not have a closed graph.

PROOF. In Example 1 of [19], $f : R \to R$ is almost continuous and R is Hausdorff. However, $G(f)$ is not closed since $(p, -p) \in Cl(G(f)) - G(f)$ for a positive integer p.

COROLLARY 6.5.4. An a.w.c. function into a Hausdorff space need not have a closed graph.

The following is shown in [23, Theorem 10].

(6.5.5) If $f : X \to Y$ is weakly continuous and Y is Hausdorff, then $G(f)$ is closed.

The above result was improved by Baker [17] as follows:

(6.5.6) If $f : X \to Y$ is s.w.c. and Y is Hausdorff, then $G(f)$ is closed.

6.6. PRESERVATIONS OF CONNECTEDNESS AND HYPERCONNECTEDNESS.

In this section we investigate if connected spaces and hyperconnected spaces are preserved under seven weak forms of continuity. A space X is said to be *hyperconnected* if every nonempty open set of X is dense in X. The following are shown in Example 2.4 and Remark 3.2 of [24] and [22, Example 3].

(6.6.1) Neither semi continuous surjections nor almost continuous surjections preserve connected spaces in general.

The following is shown in [20, Theorem 3].

(6.6.2) Weakly continuous surjections preserve connected spaces.

THEOREM 6.6.3. Connectedness is not necessarily preserved under s.w.c. surjections.

PROOF. Let X be real numbers with the finite complement topology, Y real numbers with the discrete topology and $f : X \to Y$ the identity function. Then f is a s.w.c. surjection and X is connected. However, Y is not connected.

The following is an improvement of (6.6.2) [25, Corollary 3.7].

(6.6.4) Connectedness is preserved under f.c. surjections.

COROLLARY 6.6.5. Neither w.q.c. surjections nor a.w.c. surjections preserve connected spaces in general.

PROOF. This is an immediate consequence of (6.6.1).

The following is shown in [26, Lemma 5.3].

(6.6.6) Semi continuous surjections preserve hyperconnected spaces.

THEOREM 6.6.7. Almost continuous surjections need not preserve hyperconnected spaces.

PROOF. In Example 5.8, $f : X \to Y$ is an almost continuous surjection and X is hyperconnected. However, Y is not hyperconnected.

THEOREM 6.6.8. Weakly continuous surjections need not preserve hyperconnected spaces.

PROOF. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is a weakly continuous surjection and (X, τ) is hyperconnected. However, (X, σ) is not hyperconnected.

COROLLARY 6.6.9. Hyperconnectedness is not necessarily preserved under s.w.c., f.c., w.q.c. or a.w.c. surjections.

PROOF. This follows immediately from Theorem 6.6.8.

6.7. SURJECTIONS WHICH IMPLY SET-CONNECTED FUNCTIONS.

DEFINITION 6.7.1. Let A and B be subsets of a space X. A space X is said to be *connected between A and B* if there exists no clopen set F such that $A \subset F$ and $F \cap B = \emptyset$. A function $f : X \to Y$ is said to be *set-connected* [27] provided that $f(X)$ is connected between $f(A)$ and $f(B)$ with respect to the relative topology if X is connected between A and B.

The following lemma is very useful in the sequel.

LEMMA 6.7.2 (Kwak [27]). A surjection $f : X \to Y$ is set-connected if and only if $f^{-1}(F)$ is a clopen set of X for every clopen set F of Y.

THEOREM 6.7.3. A semi continuous surjcetion need not be set-connected.

PROOF. In Example 5.2, f is a semi continuous surjection but it is not set-connected since $f^{-1}(\{a\})$ is not closed in (X, τ).

THEOREM 6.7.4. An almost continuous surjcetion need not be set-connected.

PROOF. In Example 5.8, f is an almost continuous surjection but it is not set-connected.

COROLLARY 6.7.5. Neither w.q.c. surjections nor a.w.c. surjections are set-connected in general.

PROOF. This is an immediate consequence of Theorems 6.7.3 and 6.7.4.

The following is shown in [28, Theorem 3].

(6.7.6) Every weakly continuous surjection is set-connected.

THEOREM 6.7.7. A s.w.c. surjection need not be set-connected.

PROOF. In Example 5.6, $f : (X, \tau) \to (X, \sigma)$ is a s.w.c. surjection but it is not set-connected since $f^{-1}(\{x\})$ is not open in (X, τ) for a clopen set $\{x\}$ of (X, σ).

The following is shown in [25, Theorem 3.4].

(6.7.8) Every f.c. surjection is set-connected.

7. QUESTIONS.

In this section we sum up several questions concerning subweak continuity and faint continuity.

QUESTION 1. Are the following statements for s.w.c. functions true ?

1) A function is s.w.c. if the graph function is s.w.c.

2) Each function $f_\alpha : X_\alpha \to Y_\alpha$ is s.w.c. if the product function $f : \Pi X_\alpha \to \Pi Y_\alpha$ is s.w.c.

QUESTION 2. Are the following statements for f.c. functions true ?

1) The composition of f.c. functions is f.c.

2) If a function is f.c., then the graph function is f.c.

3) If each $f_\alpha : X_\alpha \to Y_\alpha$ is f.c., then $f : \Pi X_\alpha \to \Pi Y_\alpha$ is f.c.

4) If each $q_\beta \circ f : X \to Y_\beta$ is f.c., then $f : X \to \Pi Y_\alpha$ is f.c.

5) If $f : X \to Y$ is f.c. and Y is Hausdorff, then $G(f)$ is closed in $X \times Y$.

Finally, the results obtained in Section 6 are summarized in the following table, where ( ) denotes the results already known.

TABLE

		s.c.	a.c.	w.c.	s.w.c.	f.c.	w.q.c.	a.w.c.
1	$f:X \to Y:P,\ g:Y \to Z:P$ $\to g\circ f:X \to Z:P$	(-) 6.1.1	- 6.1.2	(-) 6.1.1	(-) 6.1.1		- 6.1.3	- 6.1.3
2	$f:X \to Y:P,\ g:Y \to Z:C$ $\to g\circ f:X \to Z:P$	+ 6.1.4	+ 6.1.4	(+) 6.1.5	(+) 6.1.5	(+) 6.1.5	+ 6.1.6	+ 6.1.6
3	$f:X \to Y:C,\ g:Y \to Z:P$ $\to g\circ f:X \to Z:P$	- 6.1.7	- 6.1.8	(+) 6.1.9	+ 6.1.10	+ 6.1.10	- 6.1.7	- 6.1.8
4	$f:X \to Y:P,\ A \subset X$ $\to f\|A:A \to Y:P$	- 6.2.1	(-) 6.2.2	+ 6.2.3	(+) 6.2.4	(+) 6.2.4.	- 6.2.1	- 6.2.5
5	$f:X \to Y:P$, A:open $\to f\|A:A \to Y:P$	(+) 6.2.6	(+) 6.2.6	(+) 6.2.7	(+) 6.2.7	(+) 6.2.7	+ 6.2.8.	+ 6.2.9
6	$g:X \to X\times Y:P$ $\to f:X \to Y:P$	(+) 6.3.1	(+) 6.3.1	(+) 6.3.1		(+) 6.3.3	+ 6.3.4	+ 6.3.5
7	$f:X \to Y:P$ $\to g:X \to X\times Y:P$	(+) 6.3.1	(+) 6.3.1	(+) 6.3.1	(+) 6.3.2		+ 6.3.4	+ 6.3.5
8	$f:\Pi X_\alpha \to \Pi Y_\alpha:P$ $\to f_\alpha:X_\alpha \to Y_\alpha:P$	(+) 6.4.1	(+) 6.4.1	(+) 6.4.1		(+) 6.4.3	+ 6.4.5	+ 6.4.7
9	$f_\alpha:X_\alpha \to Y_\alpha:P$ $\to f:\Pi X_\alpha \to \Pi Y_\alpha:P$	(+) 6.4.1	(+) 6.4.1	(+) 6.4.1	(+) 6.4.2		+ 6.4.5	+ 6.4.7
10	$f:X \to \Pi Y_\alpha:P$ $\to p_\alpha\circ f:X \to Y_\alpha:P$	(+) 6.4.8	+ 6.4.9	(+) 6.4.10	+ 6.4.13	(+) 6.4.12	+ 6.4.14	+ 6.4.15
11	$p_\alpha\circ f:X \to Y_\alpha:P$ $\to f:X \to \Pi Y_\alpha:P$	(-) 6.4.8	+ 6.4.9	(+) 6.4.10	(+) 6.4.11		- 6.4.14	+ 6.4.15
12	$f:X \to Y:P,\ Y:T_2$ $\to G(f)$:closed	- 6.5.1	- 6.5.3	(+) 6.5.5	(+) 6.5.6		- 6.5.2	- 6.5.4
13	$f:X \to Y$:onto P, X:connected $\to$ Y:connected	(-) 6.6.1	(-) 6.6.1	(+) 6.6.2	- 6.6.3	(+) 6.6.4	- 6.6.5	- 6.6.5
14	$f:X \to Y$:onto P X:hyperconnected $\to$ Y:hyperconnected	(+) 6.6.6	- 6.6.7	- 6.6.8	- 6.6.9	- 6.6.9	- 6.6.9	- 6.6.9
15	$f:X \to Y$:onto P $\to$ f:set-connected	- 6.7.3	- 6.7.4	(+) 6.7.6	- 6.7.7	(+) 6.7.8	- 6.7.5	- 6.7.5

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