Internat. J. Math. & Math. Sci.
VOL. 11 NO. 1 (1988) 71-80

ON SINGULAR PROJECTIVE DEFORMATIONS OF TWO SECOND CLASS TOTALLY FOCAL PSEUDOCONGRUENCES OF PLANES

LUDMILA GOLDBERG

Department of Mathematics
New Jersey Institute of Technology
Newark, New Jersey 07102 U.S.A.

(Recieved July 31, 1986)

ABSTRACT. Let $C: L \to \overline{L}$ be a projective deformation of the second order of two totally focal pseudocongruences L and $\overline{L}$ of $(m-1)$-planes in projective spaces P^n and $\overline{P}^n$, $2m-1 \leq n < 3m-1$, and let K be a collineation realizing such a C. The deformation C is said to be weakly singular, singular, or α-strongly singular, $\alpha = 3,4,\ldots$, if the collineation K gives projective deformations of order 1, 2 or α of all corresponding focal surfaces of L and $\overline{L}$. It is proved that C is weakly singular and conditions are found for C to be singular. The pseudocongruences L and $\overline{L}$ are identical if and only if C is 3-strongly singular.

KEY WORDS AND PHRASES. *Pseudocongruence, projective deformation, singular projective deformation, focal surface.*
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 53A20

1. INTRODUCTION.

Let L and $\overline{L}$ be totally focal pseudocongruences of $(m-1)$-planes in projective spaces P^n and $\overline{P}^n$ and let $C: L \to \overline{L}$ be a correspondence between planes of L and $\overline{L}$. In the case of pseudocongruences of straight lines ($m = 2$) regular and singular projective deformations C were studied in many papers (see Svec [1] where one can find further references).

In the present paper we will suppose that $m > 2$ and $2m-1 \leq n < 3m-1$. The last restriction means that L and $\overline{L}$ are of second class, i.e. lie in their second osculating spaces provided that their first osculating spaces are tangent spaces.

The author (see Goldberg [2]) found necessary and sufficient conditions for C to be a projective deformation of order 1,2, and 3. However, conditions under which the pseudocongruences L and $\overline{L}$ are identical were not found in [2].

In the present paper we will indicate such a condition in terms of singular projective deformations. Note that second and third order singular projective deformations were studied by the author for every $n \geq 3m-1$ (see Goldberg [3]) and for every $n \geq 4m-1$ (see Goldberg [4]). Note also that second order singular projective deformations in odd-dimensional projective spaces were considered by Kreizlik [5,6].

If K is a collineation realizing a projective deformation C of second order, and at the same time K realizes projective deformations of order 1, 2, or α, $\alpha=3,4,\ldots$, of all corresponding focal surfaces of L and $\overline{L}$, then C is called weakly singular,

singular, or α-strongly singular respectively.

In the present paper it is proved that a second order projective deformation C is weakly singular, necessary and sufficient conditions for C are found to be singular, and the following condition of identity of L and $\bar{L}$ is obtained: pseudocongruences L and $\bar{L}$ related by a second order projective deformation C: $L \to \bar{L}$ are identical if and only if C is 3-strongly singular.

Note that the author proved in [2] that if $L \subset P^n$ and C: $L \to \bar{L}$ is a projective deformation, then $\bar{L} \subset \bar{P}^n$. Because of this, we suppose from the beginning that $L \subset P^n$ and $\bar{L} \subset \bar{P}^n$.

2. A SPECIALIZATION OF MOVING FRAMES ASSOCIATED WITH A TOTALLY FOCAL PSEUDOCONGRUENCE AND FUNDAMENTAL EQUATIONS.

A family L of planes of an n-dimensional projective space P^n is said to be a *pseudocongruence* if each hyperplane of P^n contains locally a unique plane of L.

A pseudocongruence L of (m - 1)-planes is a family of m parameters. The admissible m-tuples $(u_1,\dots,u_m)$ are taken from an open neighborhood of C^m (C = complex numbers).

A one-parameter subfamily of L is said to be *focal of order r* if infinitesimally close planes of L have an r-dimensional intersection. Focal subfamilies of maximum order m - 2 are called *developable surfaces* of L. A pseudocongruence of (m-1)-planes possessing the maximum number m of developable surfaces is called *focal*.

In general, (m-2)-dimensional characteristics of each of these m developable surfaces forms a symplex in a plane $p_{m-1} \in L$. The vertices of this symplex are *foci* of p_{m-1}. Each focus generates the *focal surface* of L of dimension m. A plane p_{m-1} belongs to the tangent m-plane of each of the m focal surfaces.

It was shown by Geidelman [7] that focal pseudocongruences can be of three types:

(a) Pseudocongruences whose (m-1)-planes belong to an (m+1)-plane;

(b) Pseudocongruences foliating into ∞^b subfamilies of m-b parameters, $1 \le b \le m$, where all (m-1)-planes of each of these subfamilies belong to an m-plane;

(c) Pseudocongruences possessing m systems of integrable (m-1)-parameter focal subfamilies of order zero.

Pseudocongruences of the third type are called *totally focal* (abbreviated t.f.). Each of the m focal surfaces of a t.f. pseudocongruences is an m-conjugate system (see Geidelman [7]).

Let L be a t.f. pseudocongruence of (m-1)-planes p_{m-1} in P^n. To each plane $p_{m-1} \in L$ we associate a moving frame consisting of linearly independent analytic points $A_1,\dots,A_{n+1}$, such that

$$[A_1,\dots,A_{n+1}] = 1 \tag{2.1}$$

and $(A_1,\dots,A_m) = p_{m-1}$.

The equations of infinitesimal displacements of the moving frame are

$$dA_u = \omega_u^v A_v , \qquad u,v = 1,\dots,n+1, \tag{2.2}$$

where the Pfaffian forms ω_u^v satisfy the structure equations (i.e. the integrability conditions) of the space P^n:

$$d\omega_u^v = \omega_u^w \wedge \omega_w^v , \qquad u,v,w = 1,\dots,n+1 . \tag{2.3}$$

In addition, differentiating (2.1) by means of (2.2), we obtain

$$\omega_u^u = 0 . \tag{2.4}$$

In this paper we will suppose that $2m-1 \leq n < 3m-1$. In this case we can specialize the moving frames in such a way that

(i) the vertices A_i , $i = 1,\dots,m$, are foci of p_{m-1};

(ii) the line A_iA_{m+i} is tangent to the line γ_i of the conjugate net (A_i) which is not tangent to p_{m-1};

(iii) the points $A_{2m+1},\dots,A_{2m+\sigma}$, where $2m+\sigma = n+1$, are chosen arbitrarily (of course, (2.1) is supposed to be satisfied).

Under such a choice of vertices A_u of the moving frames the developable surfaces of L are determined by equations $\omega_i^{m+i} = 0$. Since all foci are supposed to be linearly independent, forms ω_i^{m+i} are also linearly independent. We will take them as forms of the dual cobasis and will denote them by ω^i:

$$\omega_i^{m+i} = \omega^i \ . \tag{2.5}$$

In (2.5) and in what follows there is no summation of the indices $i,j,k = 1,\dots,m$, unless it is indicated by the summation sign. If the moving frames are specialized in the above described manner, we have:

$$dA_i = \omega_i^k A_k + \omega^i A_{m+i} \ , \tag{2.6}$$

$$\omega_i^{m+j} = 0 \ , \qquad j \neq i \ , \tag{2.7}$$

$$\omega_i^{2m+r} = 0 \ , \qquad r = 1,\dots,\sigma. \tag{2.8}$$

In addition, since

$$(dA_i, A_i, A_{m+i}) \equiv 0 \bmod \omega^j \ , \qquad j \neq i \ , \tag{2.9}$$

by means of (2.6) we obtain

$$\omega_i^j = b_i^j \omega^j \ , \qquad j \neq i \ . \tag{2.10}$$

Exterior differentiation of (2.7) and (2.8) by means of (2.2) and (2.10) and application of Cartan's lemma leads to

$$\omega_{m+i}^{m+j} = c_i^j \omega^i \ , \qquad j \neq i \ , \tag{2.11}$$

$$\omega_{m+i}^{2m+r} = a_i^{2m+r} \omega^i \ , \qquad r = 1,\dots,\sigma. \tag{2.12}$$

It follows from (2.1) and (2.12) that

$$dA_{m+i} = \sum_k (\omega_{m+i}^k A_k + \omega_{m+i}^{m+k} A_{m+k}) + a_i^{2m+r} \omega^i A_{2m+r} \ . \tag{2.13}$$

Since $n = 2m-1+\sigma$, there are σ linearly independent points among points $a_i^{2m+r} A_{2m+r}$. Because of this, we have

$$\operatorname{rank}(a_i^{2m+r}) = \sigma \ . \tag{2.14}$$

In other words, (2.14) means that the second osculating space of L is the whole space P^n, i.e. L is of second class.

Further exterior differentiation of (2.10), (2.11), and (2.12) and application of Cartan's lemma give rise to the following Pfaffian equations:

$$db_i^j + b_i^j(2\omega_j^j - \omega_i^i - \omega_{m+j}^{m+j}) - \sum_{k\neq i,j} b_i^k b_k^j \omega^k = b_{ij}^j \omega^j + a_{ij}^j \omega^i \ , \tag{2.15}$$

$$\omega_{m+i}^j = a_{ij}^j \omega^j - a_{ii}^j \omega^i \ , \tag{2.16}$$

$$dc_i^j + c_i^j(\omega_i^i - 2\omega_{m+i}^{m+i} + \omega_{m+j}^{m+j}) + \sum_{k\neq i,j} c_i^k c_k^j \omega^k + a_i^{2m+s}\omega_{2m+s}^{m+j} = a_{ii}^j\omega^j + c_{ii}^j\omega^i, \quad (2.17)$$

$$da_i^{2m+r} + a_i^{2m+r}(\omega_i^i - 2\omega_{m+i}^{m+i} + \omega_{2m+r}^{2m+r}) + a_i^{2m+s}\omega_{2m+s}^{2m+r} + \sum_{k\neq i} c_i^k a_k^{2m+r}\omega^k = a_{ii}^{2m+r}\omega^i. \quad (2.18)$$

In the following we will need the differential extensions of (2.15) and (2.16) which have the following form:

$$db_{ij}^j + b_{ij}^j(3\omega_j^j - \omega_i^i - 2\omega_{m+j}^{m+j}) + b_i^j(3\omega_{m+j}^j - a_j^{2m+r}\omega_{2m+r}^{m+j}) - \sum_{k\neq i,j} b_i^k b_{kj}^j\omega^k - 3b_j^i(b_i^j)^2\omega^i$$
$$+ 3b_i^j \sum_{k\neq i,j} b_j^k b_k^j\omega^k - b_i^j \sum_{k\neq i,j} c_j^k c_k^j\omega^k - b_i^j c_j^i c_i^j\omega^i = b_{ijj}^j\omega^j + a_{ijj}^j\omega^i, \quad (2.19)$$

$$da_{jj}^j + a_{ij}^j(2\omega_j^j - \omega_{m+i}^{m+i} - \omega_{m+j}^{m+j}) - b_i^j\omega_{m+i}^i + \sum_{k\neq i,j}(a_{ii}^k b_k^j - c_i^k a_{kj}^j)\omega^i - \sum_{k\neq i,j} b_k^j a_{ik}^k\omega^k$$
$$= a_{ijj}^i\omega^j - a_{iij}^j\omega^i, \quad (2.20)$$

$$da_{ii}^j + a_{ii}^j(\omega_i^i - 2\omega_{m+i}^{m+i} + \omega_j^j) - c_i^j\omega_{m+j}^j + \sum_{k\neq i,j} c_i^k a_{kk}^j\omega^k - a_i^{2m+r}\omega_{2m+r}^j$$
$$= a_{iij}^j\omega^j - c_{iij}^j\omega^i. \quad (2.21)$$

3. FIRST AND SECOND ORDER PROJECTIVE DEFORMATIONS OF T.F. PSEUDOCONGRUENCES.

It is well known that (m-1)-planes p_{m-1} of the space P^n can be represented as points of the Grassmannian $G(m-1,n)$, $\dim G = m(n-m+1)$, in a projective space $g(P^n) = P^N$ of dimension $N = \binom{n+1}{m} - 1$. Denote by $[M_1,\dots,M_m]$ Grassmann coordinates of the plane $(M_1,\dots,M_m)$. If $\{A_u\}$ is a moving frame in P^n, then $\{[A_{u_1},\dots,A_{u_m}]\}$ is a moving frame of P^N.

Let P^n and $\overline{P}^n$ be two n-dimensional projective spaces with moving frames $\{A_u\}$ and $\{\overline{A}_u\}$ and $K: P^n \to \overline{P}^n$ be a collineation given by

$$KA_u = \alpha_u^v\overline{A}_v, \qquad \det(\alpha_u^v) \neq 0. \quad (3.1)$$

The collineation K induces the collineation $g(K): g(P^n) \to g(\overline{P}^n)$ given by

$$K[A_{u_1},\dots,A_{u_m}] = \alpha_{u_1}^{v_1}\dots\alpha_{u_m}^{v_m}[\overline{A}_{v_1}\dots\overline{A}_{v_m}]. \quad (3.2)$$

A pseudocongruence L is represented in $g(P^n)$ by some surface belonging to $G(m-1,n)$. We will denote it also by L.

A correspondence $C: L \to \overline{L}$ between two t.f. pseudocongruences L and $\overline{L}$ of P^n and $\overline{P}^n$ is said to be a *projective deformation of order h* if for any plane $p_{m-1} \in L$ there exists a collineation $K: P^n \to \overline{P}^n$ such that surfaces $g(K)g(L)$ and $g(\overline{L})$ have the analytic contact of order h at the point $g(p_{m-1})$, i.e. if

$$Kd^s[A_1\dots A_m] = \sum_{\ell=0}^{s}\binom{h}{\ell}\theta_\ell\, d^{h-\ell}[\overline{A}_1\dots\overline{A}_m] \quad (3.3)$$

where $s = 0,1,\dots,h$ and θ_ℓ are ℓ-forms.

Suppose that the moving frames $\{\overline{A}_u\}$ associated to the planes $\overline{p}_{m-1} \in \overline{L}$ are specialized similarly to the moving frames associated to the planes $p_{m-1} \in L$. We will denote all expressions connected with $\overline{L}$ by suppressing the overbar. Then we have equations $\overline{(2.1)}$-$\overline{(2.21)}$ if $2m-1 \leq n < 3m-1$.

According to (3.3), the correspondence $C: L \to \overline{L}$ is a projective deformation of order one if for any $p_{m-1} \in L$ there exists a collineation $K: P^n \to \overline{P}^n$ such that

$$K[A_1 \dots A_m] = [\bar{A}_1 \dots \bar{A}_m] ,$$
$$Kd[A_1 \dots A_m] = d[\bar{A}_1 \dots \bar{A}_m] + \theta_1 [\bar{A}_1 \dots \bar{A}_m]. \tag{3.4}$$

In what follows we will denote the Grassmann products $[A_1 \dots A_m]$, $[A_1 \dots A_{k-1} A_{u_k} A_{k+1} \dots A_m]$, $[A_1 \dots A_{k-1} A_{u_k} A_{k+1} \dots A_{\ell-1} A_{u_\ell} A_{\ell+1} \dots A_m]$, ... by $[A]$, $[A^k_{u_k}]$, $[A^{k,\ell}_{u_k,u_\ell}]$, ...:

$$\left.\begin{aligned} &[A_1 \dots A_m] = [A] , \quad [A_1 \dots A_{k-1} A_{u_k} A_{k+1} \dots A_m] = [A^k_{u_k}], \\ &[A_1 \dots A_{k-1} A_{u_k} A_{k+1} \dots A_{\ell-1} A_{u_\ell} A_{\ell+1} \dots A_m] = [A^{k,\ell}_{u_k,u_\ell}], \dots \end{aligned}\right\} \tag{3.5}$$

Using notations (3.5), we can write (3.4) in the form

$$K[A] = [\bar{A}] , \qquad Kd[A] = d[\bar{A}] + \theta_1 [\bar{A}] . \tag{3.6}$$

By means of (2.1), (2.3), and (2.4) one obtains

$$d[A] = \sum_i \omega^i_i [A] + \sum_i \omega^i [A^i_{m+i}] . \tag{3.7}$$

The author proved in [4] the following theorem:

THEOREM 1. A correspondence $C: L \to \bar{L}$ is a projective deformation of first order if and only if C is developable (i.e. developable surfaces of L and $\bar{L}$ correspond to each other under C). A collineation K realizing such a deformation is determined by

$$\left.\begin{aligned} &KA_i = \rho_i \bar{A}_i , \qquad KA_{m+i} = \rho_i \bar{A}_{m+i} + \sum_j \alpha^j_{m+i} \bar{A}_j , \\ &KA_{2m+r} = \alpha^u_{2m+r} \bar{A}_u , \quad r = 1,\dots,\sigma, \quad u = 1,\dots,n+1. \end{aligned}\right\} \tag{3.8}$$

Although restrictions for n are different in [4] and in the present paper ($n \geq 4m-1$ and $2m-1 \leq n < 3m-1$ respectively), in the proof presented in [4] one needs to have $n \geq 2m-1$ only.

Note that in the proof as consequences of (3.6) the author obtained the form (3.8) of the collineation K, equalities

$$\omega^i = \bar{\omega}^i \tag{3.9}$$

giving developability of C, and the following form for the 1-form θ_1 in (3.6):

$$\theta_1 = \sum_i (- \tau^i_i + \alpha^i_{m+i} \rho_i^{-1} \omega^i) . \tag{3.10}$$

In (3.10) and what follows we use the notation

$$\tau^v_u = \bar{\omega}^v_u - \omega^v_u . \tag{3.11}$$

In addition, in what follows we will need the differential extension of equation (3.9) that has the following form:

$$\tau^{m+i}_{m+i} - \tau^i_i = t_i \omega^i . \tag{3.12}$$

In the case of a projective deformation of second order one obtains from (3.3) conditions (3.6) and

$$Kd^2[A] = d^2[\bar{A}] + 2\theta_1 d[\bar{A}] + \theta_2 [\bar{A}] . \tag{3.13}$$

Differentiation of (3.7) gives

$$d^2[A] = \sum_i [d\omega_i^i + (\omega_i^i)^2 + \omega^i\omega_{m+i}^i][A] + \sum_i [d\omega^i + \omega^i(2\sum_j \omega_j^j - \omega_i^i + \omega_{m+i}^{m+i})][A_{m+i}^i]$$
$$+ \sum_{i,j\ (i\neq j)} \left\{[c_i^j(\omega^i)^2 - b_i^j(\omega^j)^2][A_{m+j}^i] + \omega^i\omega^j[A_{m+i,m+j}^{i,\ j}]\right\}.$$
$$+ a_i^{2m+r}(\omega^i)^2[A_{2m+r}^i]) \ . \tag{3.14}$$

Note that equation (3.14) is slightly different from similar equation in the above mentioned paper [4] because of different restrictions for n and different choice of moving frames.

THEOREM 2. A correspondence $C: L \to \overline{L}$ is a projective deformation of second order if and only if there exist functions ρ_i such that the relative invariants b_i^j and $\overline{b}_i^j$ of L and $\overline{L}$ satisfy equation

$$\rho_i\overline{b}_i^j = \rho_j b_i^j \ . \tag{3.15}$$

A collineation K realizing such a deformation C is determined by (2.8) where ρ_i and α_u^v satisfy equations:

$$\left.\begin{array}{ll} \rho_1 \cdots \rho_m = 1 \ , & \rho_i\overline{c}_i^k - \rho_k c_i^k = a_i^{2m+s}\ \alpha_{2m+s}^{m+k} \ , \\ \rho_i t_i = a_i^{2m+s}\ \alpha_{2m+s}^{m+i} - 2\alpha_{m+i}^i \ . & \end{array}\right\} \tag{3.16}$$

Proof of Theorem 2 is computational and follows proof of a similar theorem in [4] where one should use (3.13) and (3.14).

Note also that in [4] the author proved that if $C: L \to \overline{L}$ is a projective deformation of second order, then the following identities hold:

$$\rho_i\overline{b}_{ij}^j = \rho_j b_{ij}^j - b_i^j(\rho_j t_j - \alpha_{m+j}^j) \ , \tag{3.17}$$
$$\rho_i\overline{a}_{ij}^j = \rho_j a_{ij}^j - \overline{b}_i^j\alpha_{m+i}^i \ . \tag{3.18}$$

They can be obtained from (3.15) and (2.15).

4. SINGULAR PROJECTIVE DEFORMATIONS OF T.F. PSEUDOCONGRUENCES.

A correspondence $C: L \to \overline{L}$ induces the correspondences $C_i: (A_i) \to (\overline{A}_i)$ of focal surfaces of L and $\overline{L}$. Suppose that there exists a collineation H such that

$$Hd^sA_i = \sum_{\ell=0}^{s} \binom{h}{\ell} \phi_i^\ell\ d^{h-\ell}\overline{A}_i \ , \qquad s = 0,1,\ldots,h, \tag{4.1}$$

where ϕ_i^ℓ are ℓ-forms. In this case we will say that C_i is *a projective deformation of order h* between (A_i) and $(\overline{A}_i)$. A second order projective deformation $C: L \to \overline{L}$ realized by the collineation K which is determined by (3.8) is said to be *weakly singular, singular,* or *α-strongly singular,* $\alpha = 3,4,\ldots$, if the correspondences C_i induced by C are projective deformations of order one, two or α respectively realized by the same collineation K.

THEOREM 3. A second order projective deformation $C: L \to \overline{L}$ is weakly singular.

PROOF. Suppose that $C: L \to \overline{L}$ is a projective deformation of second order, i.e. we have (3.8) and (3.15)-(3.18). It follows from (2.6) and (2.10) that

$$dA_i = \omega_i^i A_i + \sum_{j\neq i} b_i^j\omega^j A_j + \omega^i A_{m+i} \ . \tag{4.2}$$

Using (3.8), one finds from (4.2) that

$$KA_i = \rho_i\overline{A}_i \ , \quad KdA_i = \rho_i d\overline{A}_i + (-\rho_i\tau_i^i + \alpha_{m+i}^i\omega^i)\overline{A}_i \ . \tag{4.3}$$

According to (4.1), it means that K realizes a projective deformation of first order of (A_i) and $(\bar{A}_i)$ for any i and therefore the correspondence C: $L \to \bar{L}$ is weakly singular. Note also that (4.3) and (4.1) show that

$$\phi_i^0 = \rho_i \ , \qquad \phi_i^1 = -\rho_i \tau_i^i + \alpha_{m+i}^i \omega^i \ . \tag{4.4}$$

THEOREM 4. A second order projective deformation C: $L \to \bar{L}$ is singular if and only if the following equations hold:

$$a_i^{2m+r} \alpha_{2m+r}^j = a_{ii}^j \rho_j - \bar{a}_{ii}^j \rho_i - c_i^j \alpha_{m+j}^j \ , \tag{4.5}$$

$$\rho_i \bar{a}_i^{2m+r} = a_i^{2m+s} \alpha_{2m+s}^{2m+r} \ . \tag{4.6}$$

PROOF. Suppose again that C: $L \to \bar{L}$ is a second order projective deformation. Differentiating (4.2) and using (2.1), (2.7), (2.8), (2.11), (2.12), (2.15), and (2.16), we get

$$\begin{aligned} d^2 A_i = {} & \{d\omega_i^i + (\omega_i^i)^2 + \omega^i [\omega_{m+i}^i + \sum_{j \neq i} b_i^j b_j^i \omega^j]\} A_i + \sum_{j \neq i} [b_i^j d\omega^j + b_i^j \omega^j (2\omega_i^i + \omega_{m+j}^{m+j} - \omega_j^j) - a_{ii}^j (\omega^i)^2 \\ & + \omega^j (2 \sum_{k \neq i,j} b_i^k b_k^j \omega^k + b_{ij}^j \omega^j + 2a_{ij}^j \omega^i)] A_j + [d\omega^i + \omega^i (\omega_i^i + \omega_{m+i}^{m+i})] A_{m+i} \\ & + \sum_{j \neq i} [b_i^j (\omega^j)^2 + c_i^j (\omega^i)^2] A_{m+j} + a_i^{2m+r} (\omega^i)^2 A_{2m+r} \ . \end{aligned} \tag{4.7}$$

Using (3.11), (4.7), $(\overline{4.7})$, (3.8), (4.3), $(\overline{4.2})$, (4.5), (4.6), (3.12), and (3.15)-(3.18), we obtain

$$\begin{aligned} K d^2 A_i - \rho_i d^2 \bar{A}_i - 2\phi_i^1 d\bar{A}_i - \phi_i^2 \bar{A}_i = (\omega^i)^2 \Big[& \sum_{k \neq i} (a_i^{2m+r} \alpha_{2m+r}^k + \bar{a}_{ii}^k \rho_i - a_{ii}^k \rho_k + c_i^k \alpha_{m+k}^k) \bar{A}_k \\ & + (a_i^{2m+s} \alpha_{2m+s}^{2m+r} - \rho_i \bar{a}_i^{2m+r}) \bar{A}_{2m+r} \Big] \end{aligned} \tag{4.8}$$

where

$$\phi_i^2 = -\rho_i [d\tau_i^i - (\tau_i^i)^2 + \omega^i \tau_{m+i}^i] + \alpha_{m+i}^i [d\omega^i + \omega^i (\omega_{m+i}^{m+i} - 2\tau_i^i - \omega_i^i)] + a_i^{2m+r} \alpha_{2m+r}^i (\omega^i)^2 . \tag{4.9}$$

Comparison of (4.1) for s = 2 and (4.8) leads to equations (4.5) and (4.6). Q.E.D.

THEOREM 5. Suppose that L and $\bar{L}$ are second class t.f. pseudocongruences of planes in projective spaces P^n and $\bar{P}^n$, $2m-1 \leq n < 3m-1$, and suppose that they are related by a second order projective deformation C: $L \to \bar{L}$. The pseudocongruences L and $\bar{L}$ are identical if and only if the deformation C is 3-strongly singular.

PROOF. Suppose again that C: $L \to \bar{L}$ is a second order projective deformation between L and $\bar{L}$. The deformation C is 3-strongly singular if (4.1) holds for s = 1,2,3. We already showed that for s = 1 equation (4.1) holds automatically and for s = 2 it holds if and only if conditions (4.5) and (4.6) are satisfied.

For a 3-strongly singular deformation C we additionally have

$$K\, d^3 A_i = \rho_i\, d^3 \bar{A}_i + 3\phi_i^1\, d^2 \bar{A}_i + 3\phi_i^2\, d\bar{A}_i + \phi_i^3 \bar{A}_i \tag{4.10}$$

where ϕ_i^1 and ϕ_i^2 are 1- and 2-forms determined by (4.4) and (4.9) and ϕ_i^3 is a 3-form.

Differentiating (4.7) and using (2.1), (2.7), (2.8), (2.11), and (2.15)-(2.21), we obtain

$$d^3 A_i = (\cdot) A_i + \sum_j (\psi_i^j A_j + \psi_i^{m+j} A_{m+j}) + \psi_i^{2m+r} A_{2m+r} \tag{4.11}$$

where for $j \neq i$:

$$\left.\begin{aligned}
\psi_i^j &= 2\omega^i\omega^j(a_{ijj}^j\omega^j - a_{iij}^j\omega^i) + (\omega^j)^2(b_{ijj}^j\omega^j + a_{ijj}^j\omega^i) - (\omega^i)^2(a_{iij}^j\omega^j - c_{iij}^j\omega^j) + \beta_i^j ,\\
\psi_i^{m+i} &= \omega_{m+i}^i\omega^i + \beta_i^{m+i} ,\\
\psi_i^{m+j} &= c_{ii}^j(\omega^i)^3 + \beta_i^{m+j} ,\\
\psi_i^{2m+r} &= a_{ii}^{2m+r}(\omega^i)^3 - a_i^{2m+r}\,\omega_{2m+r}^{2m+r}(\omega^i)^2 + \beta_i^{2m+r} .
\end{aligned}\right\} \quad (4.12)$$

In (4.12) we denoted by β_u^v 2-forms which produce terms vanishing on our final step because of conditions imposed by a second order singular projective deformation C and the first conditions following from (4.11) which we are going to obtain on our final step.

Before making the final step of our proof let us simplify at first a collineation K. By a suitable choice of local frames we obtain

$$KA_u = \overline{A}_u . \quad (4.13)$$

Equation (4.13) means that

$$\alpha_u^v = \delta_u^v . \quad (4.14)$$

It follows from (4.14), (3.9)-(3.11), (3.15)-(3.18), (4.5), (4.6), (2.5), (2.10), (2.7), (2.8), (2.11), (2.12), (2.16), (4.4), and (4.9) that

$$b_i^j = \overline{b}_i^j , \quad c_i^j = \overline{c}_i^j , \quad a_{ij}^j = \overline{a}_{ij}^j , \quad b_{ij}^j = \overline{b}_{ij}^j , \quad a_{ii}^j = \overline{a}_{ii}^j , \quad a_i^{2m+r} = \overline{a}_i^{2m+r} \quad (4.15)$$

$$\tau_i^{m+i} = \tau_i^j = \tau_i^{m+j} = \tau_i^{2m+r} = \tau_{m+i}^j = \tau_{m+i}^{m+j} = \tau_{m+i}^{2m+r} = 0 , \qquad j \neq i , \quad (4.16)$$

$$\phi_i^1 = -\tau_i^i , \quad \phi_i^2 = -d\tau_i^i + (\tau_i^i)^2 - \tau_{m+i}^i\omega^i . \quad (4.17)$$

In addition, equations (3.16), (3.12), and equation (4.5) of [4] imply

$$\tau_i^i = \tau_j^j = \tau_{m+i}^{m+i} = \tau_{m+j}^{m+j} , \qquad j \neq i . \quad (4.18)$$

It follows from (2.4) and (4.18) that

$$2m\tau_i^i + \tau_{2m+r}^{2m+r} = 0 \quad \text{(no summation)}. \quad (4.19)$$

Applying (4.13) to (4.11) and using $(\overline{4.13})$, $(\overline{4.7})$, $(\overline{4.2})$, (4.17), and conditions of singularity of C, we get

$$Kd^3A_i - (d^3\overline{A}_i + 3\phi_i^1 d^2\overline{A}_i + 3\phi_i^2 d\overline{A}_i + \phi_i^3\overline{A}_i) = \sum_{j\neq i} \Omega_i^j\overline{A}_j + \sum_j \Omega_i^{m+j}\overline{A}_{m+j} + \Omega_i^{2m+r}\overline{A}_{2m+r} \quad (4.20)$$

where for $j \neq i$:

$$\left.\begin{aligned}
\Omega_i^j &= (b_{ijj}^j - \overline{b}_{ijj}^j)(\omega^j)^3 + (c_{iij}^j - \overline{c}_{iij}^j)(\omega^i)^3 + 3\omega^i\omega^j[(a_{ijj}^j - \overline{a}_{ijj}^j)\omega^j\\
&\qquad - (a_{iij}^j - \overline{a}_{iij}^j)\omega^i] + \gamma_i^j ,\\
\Omega_i^{m+i} &= 2\tau_{m+i}^i(\omega^i)^2 + \gamma_i^{m+i} ,\\
\Omega_i^{m+j} &= (c_{ii}^j - \overline{c}_{ii}^j)(\omega^i)^3 ,\\
\Omega_i^{2m+r} &= a_i^{2m+r}\,\tau_{2m+r}^{2m+r}(\omega^i)^2 + (a_{ii}^{2m+r} - \overline{a}_{ii}^{2m+r})(\omega^i)^3 ,
\end{aligned}\right\} \quad (4.21)$$

and γ_u^v in (4.21) are 3-forms vanishing on our final step.

Comparison of (4.11) and (4.20) leads us to the following conclusions:

i) First of all, we get from the comparison that

$$\tau^{2m+r}_{2m+r} = (\bar{a}^{2m+r}_{ii} - a^{2m+r}_{ii})\ (a^{2m+r}_{i})^{-1}\ \omega^{i} \quad \text{(no summation)}.$$

Since ω^{i} are linearly independent, it implies

$$\tau^{2m+r}_{2m+r} = 0 \qquad \text{(no summation)}\ , \tag{4.22}$$

$$a^{2m+r}_{ii} = \bar{a}^{2m+r}_{ii}\ . \tag{4.23}$$

Equations (4.19), (4.22), and (4.18) give

$$\tau^{i}_{i} = \tau^{m+i}_{m+i} = 0\ . \tag{4.24}$$

It follows from (4.23), (2.18), and $(\overline{2.18})$ that

$$a^{2m+s}_{i}\ \tau^{2m+r}_{2m+s} = 0\ . \tag{4.25}$$

ii) Second of all, the comparison gives

$$c^{j}_{ii} = \bar{c}^{j}_{ii}\ . \tag{4.26}$$

Equations (4.26), (4.15), (2.17), and $(\overline{2.17})$ leads to

$$a^{2m+s}_{i}\ \tau^{j}_{2m+s} = 0\ . \tag{4.27}$$

iii) Further, we obtain from the comparison that

$$\tau^{i}_{m+i} = 0\ . \tag{4.28}$$

Note that $\gamma^{m+i}_{i} = 0$ by means of (4.24).

iv) Finally, the comparison gives

$$a^{j}_{iij} = \bar{a}^{j}_{iij}\ ,\quad a^{j}_{ijj} = \bar{a}^{j}_{ijj}\ ,\quad b^{j}_{ijj} = \bar{b}^{j}_{ijj}\ ,\quad c^{j}_{iij} = \bar{c}^{j}_{iij}\ . \tag{4.29}$$

Note that $\gamma^{j}_{i} = 0$ by means of (4.24) and other conditions which were previously obtained.

It follows from (4.29), (4.15), (2.21), and $(\overline{2.21})$ that

$$a^{2m+s}_{i}\ \tau^{j}_{2m+s} = 0\ . \tag{4.30}$$

Equality (2.14) means that the rank of the matrix of coefficients of each of the linear homogeneous systems (4.25), (4.27), and (4.30) is maximal and equal to the number of unknowns. Therefore, these systems lead to

$$\tau^{j}_{2m+r} = \tau^{m+j}_{2m+r} = \tau^{2m+s}_{2m+r} = 0\ . \tag{4.31}$$

Equations (4.16), (4.22), (4.24), (4.28), and (4.31) show that all forms $\tau^{v}_{u} = 0$. Therefore, the pseudocongruences L and $\bar{L}$ are identical. The converse statement is trivial. Q.E.D.

REFERENCES

1. SVEC, A. Projective Differential Geometry of Line Congruences, Czechoslovak Academy of Sciences, Prague, 1965.

2. GOLDBERG, L. G. (Pikuleva, L. G.) A projective deformation of totally focal pseudocongruences of planes in multidimensional projective spaces (Russian), Kalinin Gos. Ped. Inst. Uchen. Zap., 74 (1970) 111-123.

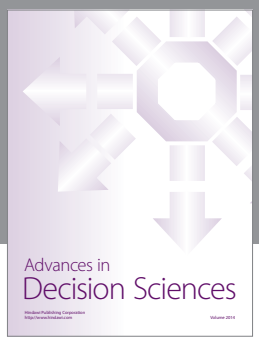
3. GOLDBERG, L. G. (Pikuleva, L. G.) A singular projective deformation of totally focal pseudocongruences of planes in multidimensional projective spaces (Russian), Akad. Nauk Armyan SSR Dokl. 60 (1975), No. 5, 257-262.

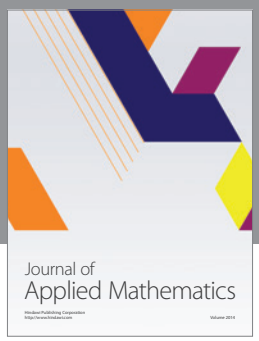
4. GOLDBERG, L. G. The third order singular projective deformations of totally focal pseudocongruences of planes in multidimensional projective spaces, Atti Accad. Peloritana Pericolanti 62 (1984), 45-72 (1986).

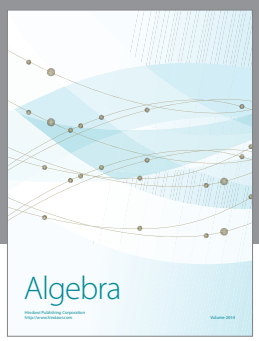
5. KREIZLIK, J. Deformations of plane pseudocongruences with projective connection, Czech. Math. J. 21 (96) (1971) 213-233.

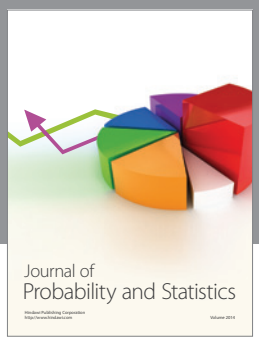
6. KREIZLIK, J. Contribution to the theory of pseudocongruences with projective connection, Arch. Math. (Brno) 17 (1981), no. 1, 31-41.

7. GEIDELMAN, R. M. A theory of pseudocongruences and congruences of planes of multidimensional hyperbolic space and congruences of spheres of multidimensional conformal space (Russian), Mat. Sb. 36 (78) (1955), No. 2, 209-232.









Hindawi

Submit your manuscripts at
http://www.hindawi.com