MULTIPLIERS ON WEIGHTED HARDY SPACES OVER CERTAIN TOTALLY DISCONNECTED GROUPS

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ABSTRACT. In this note, we consider the multipliers on weighted H^1 spaces over totally disconnected locally compact abelian groups with a suitable sequence of open compact subgroups (Vilenkin groups). We first show an (H^1, L^1) multiplier result from which Onneweer's theorem follows. We also give an (H^1, H^1) multiplier result under a condition of Baernstein-Sawyer type.

KEY WORDS AND PHRASES. Totally disconnected groups, Weighted H^1 spaces, Weighted L^p spaces, Multipliers.

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1. INTRODUCTION.

Recently, Onneweer obtained a weighted L^p multiplier theorem [1, Theorem 1] over a Vilenkin group which is a generalization of Taibleson's theorem over a local field.

In this note, we show a weighted (H^1, L^1) multiplier theorem under a weaker hypothesis than [1, Proposition 2], and show the Onneweer's theorem, by using an extended interpolation theorem for weighted H^1 and L^p spaces. We do not know whether this multiplier is also a weighted (H^1, H^1) multiplier. But we are able to show that a Baernstein-Sawyer type condition [2] which is stronger than Onneweer's, implies a weighted (H^1, H^1) result. This is also a generalization of Theorem 2 [2].

2. DEFINITIONS AND NOTATIONS.

Throughout this note, G will denote a locally compact abelian group with a sequence $\{G_n\}_{n=-\infty}^{\infty}$ such that

(i) each G_n is an open compact subgroup of G ,

(ii) $G_{n+1} \subsetneq G_n$ and $\text{order}(G_n/G_{n+1}) < \infty$,

(iii) $\bigcup_{n=-\infty}^{\infty} G_n = G$ and $\bigcap_{n=-\infty}^{\infty} G_n = \{0\}$.

Moreover we shall assume that G is order-bounded, i.e.;

$$B := \sup \{ \text{order } (G_n/G_{n+1}); n \in \mathbb{Z} \} < \infty.$$

Let Γ denote the dual group of G and for each $n \in \mathbb{Z}$, let Γ_n denote the annihilator of G_n . Then we have

(i)' each Γ_n is an open compact subgroup of Γ ,

(ii)' $\Gamma_n \subsetneq \Gamma_{n+1}$ and $\text{order } (\Gamma_{n+1}/\Gamma_n) = \text{order } (G_n/G_{n+1})$,

(iii)' $\bigcup_{-\infty}^{\infty} \Gamma_n = \Gamma$ and $\bigcap_{-\infty}^{\infty} \Gamma_n = \{1\}$.

We choose Haar measures μ on G and λ on Γ so that $\mu(G_0) = \lambda(\Gamma_0) = 1$, then $\mu(G_n) = (\lambda(\Gamma_n))^{-1} = (m_n)^{-1}$ for each $n \in \mathbb{Z}$. For an arbitrary set A we denote its indicator function by ξ_A . The symbols $\wedge$ and $\vee$ will be used to denote the Fourier and inverse Fourier transform respectively. It is easy to see that for each $n \in \mathbb{Z}$ we have $(\xi_{G_n})^\wedge = (\lambda(\Gamma_n))^{-1} \xi_{\Gamma_n}$. We set $D_n := (\mu(G_n))^{-1} \xi_{G_n}$ for each $n \in \mathbb{Z}$.

We now define the weighted L^p spaces. For $\alpha \in \mathbb{R}$, we define the function v_α on G by $v_\alpha(x) = (m_n)^{-\alpha}$ if $x \in G_n \setminus G_{n+1}$ ($n \in \mathbb{Z}$); $= 0$ if $x = 0$. We denote the L^p spaces with respect to the measure $d\mu_\alpha := v_\alpha d\mu$ on G by $L_\alpha^p(G)$, simply L_α^p . Also for $1 < p < \infty$, we set

$$\|f\|_{p,\alpha} := \left(\int_G |f(x)|^p d\mu_\alpha \right)^{1/p}.$$

Let $S(G)$ be the set of all functions ϕ on G such that ϕ has compact support and is constant on the cosets of some subgroup G_n (n depends on ϕ) of G . The functions in $S(G)$ are called test functions on G . It is well known that if $\alpha > -1$, then $S(G)$ is dense in L_α^p for $1 < p < \infty$.

In order to define the weighted Hardy spaces on G , we first define weighted atoms on G . Let $1 < q < \infty$. A function $a(x)$ on G is a $(1, q)_\alpha$ atom if there exists an interval $I = I_n(x) := x + G_n$, $x \in G$, $n \in \mathbb{Z}$ such that

(i) $\text{supp } a$ is contained in I ,

(ii) $\left(\frac{1}{\mu_\alpha(I)} \int_I |a(x)|^q d\mu_\alpha \right)^{1/q} < \mu_\alpha(I)^{-1}$, if $1 < q < \infty$

and $|a(x)| < \mu_\alpha(I)^{-1}$, if $q = \infty$,

(iii) $\int a(x) d\mu = 0$.

The weighted Hardy space $H_\alpha^{1,q}(G)$, simply $H_\alpha^{1,q}$, is the space of all functions f on G such that $f(x) = \sum_{k=0}^{\infty} \lambda_k a_k(x)$,

where the a_k 's are $(1, q)_\alpha$ atoms and $\sum_{k=0}^{\infty} |\lambda_k| < \infty$. We set $\|f\|_{H_\alpha^{1,q}} := \inf \sum_{k=0}^{\infty} |\lambda_k|$, where

the infimum is taken over all such decompositions. Then $H_\alpha^{1,q}$ is a subspace of L_α^1 and

a Banach space with the norm $\|\cdot\|_{H_\alpha^{1,q}}$. It also follows easily from the definition that

$$H_\alpha^{1,\infty} \subset H_\alpha^{1,q_2} \subset H_\alpha^{1,q_1}$$

whenever $1 < q_1 < q_2 < \infty$. We denote $H_\alpha^{1,\infty}$ by H_α^1 . In the following section, we show that $H_\alpha^{1,q} = H_\alpha^1$ if $-1 < \alpha < 0$ and $1 < q < \infty$.

We say that $m \in L^\infty(\Gamma)$ is an (X, Y) multiplier (or a multiplier on X , when $X = Y$) if there exists a constant $C > 0$ so that

$$\|(\widehat{m\phi})^\vee\|_Y \leq C \|\phi\|_X \quad \text{for all } \phi \in X \cap S(G)$$

where X and Y are equal to H_α^1 or L_α^p .

According to [1], we say that $\phi \in L^\infty(\Gamma)$ satisfies condition $C(k, r)$ for some $k \in \mathbb{Z}$ and $r \in [1, \infty)$ if there exist $C, \varepsilon > 0$ so that for all $\ell, n \in \mathbb{Z}$ with $n < \ell$ have

$$\begin{aligned} & \sup \left\{ \left(\int_{G_n \setminus G_{n+1}} |(\phi^k)^\vee(x-y) - (\phi^k)^\vee(x)|^r d\mu \right)^{1/r}; y \in G_\ell \right\} \\ & \leq C (m_n)^{1/r'} + \varepsilon (m_\ell)^{-\varepsilon}, \text{ if } 1 < r < \infty, \end{aligned}$$

and there exists $C > 0$ so that for all $\ell \in \mathbb{Z}$ we have

$$\sup \left\{ \int_{G \setminus G_\ell} |(\phi^k)^\vee(x-y) - (\phi^k)^\vee(x)| d\mu; y \in G_\ell \right\} \leq C, \text{ if } r = 1,$$

where $\phi^k = \phi \xi_{\Gamma_k}$ for each $k \in \mathbb{Z}$ and r' denotes the conjugate exponent of r .

Let $-\infty < \alpha < \infty$, $1 < p < \infty$ and $0 < q < \infty$. A function f on G belongs to the Herz space $K_p^{\alpha,q}(G)$, simply $K_p^{\alpha,q}$, if

$$\|f\|_{K_p^{\alpha,q}} = \left(\sum_{n=-\infty}^{\infty} \left\| (m_n)^{-\alpha} f \xi_{G_n \setminus G_{n+1}} \right\|_{p^q}^{1/q} \right)^q < \infty,$$

with the usual modification if $q = \infty$ [3].

We now state the main theorems:

THEOREM 1. Let $\phi \in L^\infty(\Gamma)$ and suppose that ϕ satisfies condition $C(k, r)$ for some $k \in \mathbb{Z}$ and $r \in [1, \infty)$. Then ϕ^k is an (H_α^1, L_α^1) multiplier for $-1/r' < \alpha < 0$.

As a Corollary we obtain Theorem 1 of [1]:

COROLLARY. Let $\phi \in L^\infty(\Gamma)$. (i) Suppose that condition $C(k, r)$ holds for all $k \in \mathbb{Z}$, for some $r \in (1, \infty)$, and with constants C and ε independent of $k \in \mathbb{Z}$. If ϕ is a multiplier on $L_{\alpha_0}^2$ for some α_0 with $-1/r' < \alpha_0 < 1/r'$, then ϕ is a multiplier on L_α^p for all p, α such that $1 < p < \infty$ and $-|\alpha_0| < \alpha < (p-1)|\alpha_0|$. (ii) If $C(k, 1)$ holds for all $k \in \mathbb{Z}$, and with C independent of $k \in \mathbb{Z}$, then ϕ is a multiplier on L^p for $1 < p < \infty$.

THEOREM 2. Let $\phi \in L^\infty(\Gamma)$ and suppose that there exist $r \in [1, \infty)$ and $\varepsilon > 0$ such that

$$\left\| \left(\phi_j \right)^\vee \right\|_{K_r^{\varepsilon+1/r', \infty}} < C (m_j)^{-\varepsilon} \text{ for all } j \in \mathbb{Z},$$

where $\phi_j := \phi \chi_{\Gamma_{j+1} \setminus \Gamma_j}$ for each $j \in \mathbb{Z}$, then ϕ is a multiplier on H_α^1 for $-1 < r'\alpha < 0$.

3. PRELIMINARY RESULTS.

To prove Theorem 2, we need the "maximal function" characterization of H_α^1 . For f locally in $L_\alpha^1(G)$ we define the maximal function $M_\alpha f$ of f by

$$M_\alpha f(x) := \sup_I \left\{ \frac{1}{\mu_\alpha(I)} \int_I |f(y)| d\mu_\alpha(y) \right\},$$

where the I 's are intervals containing x . When $\alpha = 0$, we denote M_α by M , simply.

LEMMA. Let $\alpha > -1$.

- (a) $\mu_\alpha(x+G_n) \leq C \mu_\alpha(x+G_{n+1})$ for all $x \in G$ and $n \in \mathbb{Z}$,
- (b) M_α is of weak-type $(1,1)$ on L_α^1 and is of type (p,p) on L_α^p for $1 < p < \infty$,
- (c) If $\alpha < 0$, then for all interval I

$$\mu_\alpha(I) \leq C \mu(I) \inf\{v_\alpha(y); y \in I, y \neq 0\}.$$

- (d) If $\alpha < 0$, then M is of weak-type $(1,1)$ on L_α^1 .

PROOF. (a) and (c) are Lemmas 1(b) and (c) in [1]. (b) follows from (a). By (c), we have that $Mf(x) \leq C M_\alpha f(x)$ for each $x \in G$. Then (d) follows from (b).

THEOREM A. Let $\alpha > -1$. An $f \in L_\alpha^1$ belongs to H_α^1 if and only if $f^* := Mf \in L_\alpha^1$. Moreover $\|f\|_{H_\alpha^1}$ is equivalent to $\|f^*\|_{1,\alpha}$.

A slight modification of the argument in [2] establishes the result, so we omit the proof.

THEOREM B. Let $-1 < \alpha < 0$. Then $H_\alpha^{1,q} \approx H_\alpha^1$, for $1 < q < \infty$.

PROOF. We have already seen that H_α^1 is continuously included in $H_\alpha^{1,q}$, for each $1 < q < \infty$. In order to establish the opposite inclusion, it suffices to show that a $(1,q)_\alpha$ atom a has the representation

$$a(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x) \quad (3.1)$$

where each a_j is a $(1,\infty)_\alpha$ atom and $\sum_{j=0}^{\infty} |\lambda_j| < C$, C independent of a . Like the non-weighted case, this can be done by using the Calderon-Zygmund decomposition [4], [5].

Let a be a $(1,q)_\alpha$ atom that is supported on $I: = x_0 + G_{n_0}$ ($x_0 \in G$, $n_0 \in \mathbb{Z}$). We let $b(x) := \mu_\alpha(I)a(x)$, then $\text{supp } b \subset I$, $\int b(x) d\mu(x) = 0$, and $\|b\|_{q,\alpha}^q \leq \mu_\alpha(I)$.

For $t > 0$ (we shall be explicit later), we denote the open set $\{x \in G: M_\alpha^q(b) > t\} = \{x \in G: M_\alpha(|b|^q)(x) > t^q\}$ by U_t . We note that $U_t \subset I$ for $t > 1$. (This is easily seen from the fact that for any two intervals in G , they are disjoint or one contains the other). Lemma (b) implies that

$$\mu_\alpha(U_t) \leq C \|b\|_{q,\alpha}^q / t^q \leq C \mu_\alpha(I) / t^q \quad (3.2)$$

and $\mu_\alpha(G_k) \rightarrow \infty$ as $k \rightarrow -\infty$ [1, Lemma (a)]. Thus we have the decomposition

$U_t = \bigcup_j I_j$; where the I_j 's are maximal disjoint sub-intervals of U_t . The Calderon-Zygmund decomposition is now that $b(x) = g_0(x) + \sum_j h_j$, where $g_0(x) = b(x)$ if

$$x \notin U_t; = m(b, I_j) \text{ if } x \in I_j \text{ and } h_j(x) = (b(x) - g_0(x)) \xi_{I_j}(x), \text{ and where } m(b, I_j)$$

denotes the average of b over I_j with respect to μ . Then the maximality of

the I_j 's and Lemma (a), (b) imply that $|g_0(x)| \leq C_0 t$, μ_α -a.e. and

$$\frac{1}{\mu_\alpha(I_j)} \int_{I_j} |h_j| d\mu_\alpha \leq \left(\frac{1}{\mu_\alpha(I_j)} \int_{I_j} |h_j|^q d\mu_\alpha \right)^{1/q} \leq 2C_0 t := C_1 t$$

by Lemma (c). If we set $(C_1 t)^{-1} h_j = b_j$, then b_j is supported in I_j , $\int b_j d\mu = 0$ and $\|b_j\|_{q,\alpha}^q \leq \mu_\alpha(I_j)$ for each j .

The idea will be now to do for each b_j the same kind of decomposition that we performed for b (with the same t) and to build an induction process which will eventually lead to the decomposition (3.1). We shall use multi-indices for the successive decomposition, in the following way:

$$\begin{aligned} b(x) &= g_0(x) + \sum_{j_0} h_{j_0}(x) = g_0(x) + C_1 t \sum_{j_0} b_{j_0}(x) \\ &= g_0(x) + C_1 t \sum_{j_0} (g_{j_0}(x) + \sum_{j_1} h_{j_0, j_1}(x)) \\ &= g_0(x) + C_1 t \sum_{j_0} g_{j_0}(x) + C_1 t \sum_{j_0, j_1} h_{j_0, j_1}(x) \\ &= g_0(x) + C_1 t \sum_{j_0} g_{j_0}(x) + \dots + (C_1 t)^n \sum_{j_0, \dots, j_{n-1}} g_{j_0, \dots, j_{n-1}}(x) \\ &\quad + (C_1 t)^n \sum_{j_0, \dots, j_n} h_{j_0, \dots, j_n}(x) \end{aligned} \quad (3.3)$$

for each $n \in \mathbb{N}$, where, $b_{j_0, \dots, j_{n-1}} := (C_1 t)^{-1} h_{j_0, \dots, j_{n-1}}$ and

$$(i) \quad \mu_\alpha(\{M_\alpha^q(b_{j_0, \dots, j_{n-1}}) > t\}) \leq C \mu_\alpha(I_{j_0, \dots, j_{n-1}}) / t^q$$

$$(ii) \{M_{\alpha}^q(b_{j_0, \dots, j_{n-1}}) > t\} = \bigcup_{j_n} I_{j_0, \dots, j_n}$$

$$(iii) \operatorname{supp} h_{j_0, \dots, j_n} \subset I_{j_0, \dots, j_n}, \quad \int h_{j_0, \dots, j_n} d\mu = 0$$

$$(iv) \left(\frac{1}{\mu_{\alpha}(I_{j_0, \dots, j_n})} \int_{I_{j_0, \dots, j_n}} |h_{j_0, \dots, j_n}|^q d\mu_{\alpha} \right)^{1/q} < C_1 t,$$

$$(v) |g_{j_0, \dots, j_{n-1}}(x)| < C_0 t,$$

for every $j_0, \dots, j_n$ and $n \in \mathbb{N}$.

By using (i), (ii) and (iv), we see that the L_{α}^1 -norm of the last term in the right

hand side of (3.3) is bounded by $(Ct^{1-q})^{n+1} \mu_{\alpha}(I)$. Hence for large $t > 0$ so that $Ct^{1-q} < 1$, we have that

$$\begin{aligned} b(x) &= g_0(x) + C_1 t \sum_{j_0} g_{j_0}(x) + \dots \\ &+ (C_1 t)^n \sum_{j_0, \dots, j_{n-1}} g_{j_0, \dots, j_{n-1}}(x) + \dots, \text{ in } L_{\alpha}^1. \end{aligned}$$

Let $a_0 := (C_0 t \mu_{\alpha}(I))^{-1} g_0$ and $a_{j_0, \dots, j_{n-1}} := (C_0 t)^{\mu_{\alpha}(I_{j_0, \dots, j_{n-1}})} g_{j_0, \dots, j_{n-1}}$ for

each $j_0, \dots, j_{n-1}$, $n \in \mathbb{N}$, then these are $(1, \infty)_{\alpha}$ atoms by (iii) and (v). Thus we obtain that

$$\begin{aligned} a(x) &= \mu_{\alpha}(I)^{-1} b(x) \\ &= C_0 t \mu_{\alpha}(I)^{-1} (\mu_{\alpha}(I) a_0(x) + C_1 t \sum_{j_0} \mu_{\alpha}(I_{j_0}) a_{j_0}(x) + \dots \\ &+ (C_1 t)^n \sum_{j_0, \dots, j_{n-1}} \mu_{\alpha}(I_{j_0, \dots, j_{n-1}}) a_{j_0, \dots, j_{n-1}}(x) + \dots), \end{aligned}$$

which is the desired representation (3.1). For, the sum of the absolute value of the coefficients of the right hand side is bounded by $C_0 t \sum_{k=0}^{\infty} (Ct^{1-q})^k = C$, independent of a . This completes the proof.

THEOREM C. Let $-1 < \alpha < 0$ and $1 < p_1 < \infty$. Suppose that T is a sublinear operator of weak-type $(1, 1)$ on H_{α}^1 , by which we mean that there exists B_0 such that for every $f \in H_{\alpha}^1$ and $t > 0$;

$$\mu_{\alpha}\{x \in G; |Tf(x)| > t\} \leq B_0 \|f\|_{H_{\alpha}^1} / t,$$

and T is of weak-type on $L_{\alpha}^{p_1}$ with constant B_1 . Then for $1 < p < p_1$, T is of type (p, p) on L_{α}^p with constant depending only on B_0 , B_1 , p_1 and p .

PROOF. The proof is similar to the non-weighted case [4], [5].

Let $f \in L_{\alpha}^p$ and choose a q so that $1 < q < p < p_1 < \infty$. As in the proof of Theorem B, we consider the open set $E_t := \{M_{\alpha}^q f > t\} = \{M_{\alpha}(|f|^q) > t^q\}$, for $t > 0$.

Then we have the same kind of decomposition; $E_t = \bigcup_j I_j$. From this we obtain a Calderon-Zygmund decomposition $f = g_t + h_t$, where $g_t = f$ if $x \notin E_t$; $= m(f, I_j)$ if $x \in I_j$ for each j , and $h_t = \sum_j h_j$, where $h_j := (f - g_t) \chi_{I_j}$. We then have $|g_t(x)| < C_0 t$ and

$$\left(\frac{1}{\mu_\alpha(I_j)} \int_{I_j} |h_j|^q d\mu_\alpha \right)^{1/q} < C_1 t$$

for each $j \in N$. Hence $a_j := (C_1 t \mu_\alpha(I_j))^{-1} h_j$ is a $(1, q)_\alpha$ atom and

$$h = C_1 t \sum_j \mu_\alpha(I_j) a_j \in H_\alpha^{1, q}. \quad \text{And Theorem B implies that } h \in H_\alpha^1 \text{ with norm bounded}$$

by $C t \mu_\alpha(E_t)$. The rest of proof proceed as in [4], [5] with a few modifications, so we omit the details.

4. PROOFS OF THE MAIN RESULTS.

PROOF OF THEOREM 1. Let $-1/r' < \alpha < 0$. To prove the conclusion, it suffices to show that $\|K*a\|_{1, \alpha} < C$ for every $(1, \infty)_\alpha$ atom a , where $K := (\phi^k)^\vee$. Let a be such an atom, supported on an interval $I = x_0 + G_n$ ($x_0 \in G$, $n \in \mathbb{Z}$). We write

$$\int_G |K*a| d\mu_\alpha = \int_I + \int_{G \setminus I} = A + B, \text{ say.}$$

Let first $r = 1$ (hence $\alpha = 0$). Then

$$A < \left(\int_I |K*a(x)|^2 d\mu \right)^{1/2} \left(\int_I d\mu \right)^{1/2} < C \|a\|_2 \mu(I)^{1/2} < C \mu(I)^{-1} \mu(I) = C.$$

On the other hand,

$$\begin{aligned} B &= \int_{G \setminus I} \left| \int_G K(x-y) a(y) d\mu(y) \right| d\mu(x) = \int_G \left| \int_G K(x-y) - K(x-x_0) a(y) d\mu(y) \right| d\mu(x) \\ &< \int_I |a(y)| d\mu(y) \int_{G \setminus I} |K(x-y) - K(x-x_0)| d\mu(x) \\ &= \int_{G_n} |a(x_0+y)| d\mu(y) \int_{G \setminus G_n} |K(x-y) - K(x)| d\mu(x) < C \int_G |a(x_0+y)| d\mu(y) < C. \end{aligned}$$

hence the conclusion follows, when $r = 1$. This together with Theorem C implies the conclusion of Corollary (ii).

Next, let $r > 1$. To estimate A , we use Corollary (ii) and Lemma (c). Then

$$\begin{aligned} A &< \left(\int_I |K*a(x)|^r d\mu \right)^{1/r} \left(\int_I (v_\alpha(x))^{r'} d\mu \right)^{1/r'} < C \|a\|_r \left(\int_I v_{\alpha r'}(x) d\mu \right)^{1/r'} \\ &< C \mu_\alpha(I)^{-1} \mu(I)^{1/r} \mu(I)^{1/r'} \inf\{v_\alpha(x); x \in I, x \neq 0\} < C \mu_\alpha(I)^{-1} \mu_\alpha(I) = C. \end{aligned}$$

On the other hand, using Lemma (c) again,

$$B < \int_I |a(y)| d\mu(y) \int_{G \setminus I} |K(x-y) - K(x-x_0)| v_\alpha(x) d\mu(x)$$

$$\begin{aligned}
&= \int_{G_n} |a(x_0+y)| d\mu(y) \int_{G \setminus G_n} |K(x-y)-K(x)| v_\alpha(x+x_0) d\mu(x) \\
&= \sum_{\ell=-\infty}^{n-1} \int_{G_n} |a(x_0+y)| d\mu(y) \int_{G_\ell \setminus G_{\ell+1}} |K(x-y)-K(x)| v_\alpha(x+x_0) d\mu(x) \\
&\leq \sum_{-\infty}^{n-1} \int_{G_n} |a(x_0+y)| d\mu(y) \int_{G_\ell \setminus G_{\ell+1}} |K(x-y)-K(x)|^r d\mu(x) \Big)^{1/r'} \\
&\quad \times \left(\int_{G_\ell \setminus G_{\ell+1}} v_{\alpha r'}(x+x_0) d\mu(x) \right)^{1/r'} \\
&\leq \sum_{-\infty}^{n-1} \int_{G_n} |a(x_0+y)| d\mu(x) (m_\ell)^{\varepsilon+1/r'} (m_n)^{-\varepsilon} m_\ell^{-1/r'} \\
&\quad \times \inf \{v_\alpha(x); x \in I, x \neq 0\} \\
&\leq C(m_n)^{-\varepsilon} \sum_{-\infty}^{n-1} (m_\ell)^\varepsilon \int_{G_n} |a(x_0+y)| v_\alpha(x_0+y) d\mu(y) \leq C(m_n)^{-\varepsilon} (m_{n-1})^\varepsilon \|a\|_{1,\alpha} \leq C.
\end{aligned}$$

This completes the proof.

PROOF OF COROLLARY. (i) Since $\phi \in L^\infty(\Gamma)$ is a multiplier on L^2 , it follows from a classical interpolation theorem for weighted spaces [6] and [1, Proposition 1] that ϕ is a multiplier on L_α^2 for all $-\alpha_0 < \alpha < \alpha_0$. As in the proof of [1, Theorem 1], the case where $1 < p < 2$ and $-\alpha_0 < \alpha < 0$, has to be proved.

Let $1 < p < 2$ and $-\alpha_0 < \alpha < 0$. Since each ϕ^k , $k \in \mathbb{Z}$ is a multiplier on L_α^2 and also a (H_α^1, L_α^1) multiplier by Theorem 1, it follows from Theorem C that ϕ^k is a multiplier on L_α^p . The assumption that the constants C and ε are independent of k , implies that ϕ is a multiplier on L_α^p .

(ii) This is already seen in the proof of Theorem 1.

PROOF OF THEOREM 2. According to Theorem A, it suffices to show that $\|(\check{\phi} * a)^*\|_{1,\alpha} \leq C$ for all $(1, \infty)_\alpha$ atom a . Let a be a $(1, \infty)_\alpha$ atom, supported on an interval $I := x_0 + G_n$ ($x_0 \in G$, $n \in \mathbb{Z}$). We set $\check{\phi} * a = f$. The case where $r = 1$ (hence $\alpha = 0$) is known [2, Corollary]. So we let $1 < r < \infty$ and $-1/r' < \alpha < 0$. Now we write

$$\int_G f * d\mu_\alpha = \int_I + \int_{G \setminus I} = A + B, \quad \text{say.}$$

We first estimate A . Since $K_r^{1/r' + \varepsilon, \infty} K_1^{\varepsilon, \infty}$ Lemma (b) and [2, Corollary]

imply that $\|f^*\|_r \leq C \|f\|_r \leq C \|a\|_r$. Thus as in the proof of Theorem 1, we have that

$$\begin{aligned}
A &\leq \left(\int_I (f^*)^r d\mu \right)^{1/r} \left(\int_I v_{\alpha r'} d\mu \right)^{1/r'} \leq C \|a\| \mu(I)^{1/r'} \inf \{v_\alpha(x); x \in I, x \neq 0\} \\
&\leq C \mu_\alpha(I)^{-1} \mu_\alpha(I) = C
\end{aligned} \tag{4.1}$$

Let $\psi(\gamma) := \overline{(\gamma, x_0)}\phi(\gamma)$ and $b(x) := a(x+x_0)$. Then it is easily seen

that $f = \phi * a = \psi * b$, $\text{supp } b \subset G_n$, and $\int b d\mu = 0$. Thus we have that

$b * D_k = 0$ if $k < n$, and $\text{supp } (b * D_k) \subset G_n$ if $k > n$. Also $(b * D_k)_j$:
 $= (b * D_k) * (D_{j+1} - D_j) = 0$ if $j > k$ and $(b * D_k)_j = b_j$ if $j < k$. Moreover $b_j = 0$, if $j < n$.
Hence

$$\begin{aligned} f^*(x) &= (\check{\psi} * b)^* = \sup_k |(\check{\psi} * b) * D_k(x)| = \sup_{k > n} |\check{\psi} * (b * D_k)(x)| \\ &= \sup_{k > n} \left| \sum_{j=-\infty}^{\infty} (\check{\psi}_j)^* (b * D_k)_j(x) \right| = \sup_{k > n} \left| \sum_{j=-\infty}^{k-1} (\check{\psi}_j)^* b_j(x) \right| < \sum_{j=-\infty}^{\infty} |(\check{\psi}_j)^* b_j(x)|. \end{aligned}$$

Then,

$$B = \int_{G \setminus I} f^* d\mu_\alpha < \sum_{j=-\infty}^{\infty} \int_{G \setminus I} |(\check{\psi}_j)^* b| d\mu_\alpha = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{n-1} \int_{I_i \setminus I_{i+1}} |(\check{\psi}_j)^* b| d\mu_\alpha, \quad (4.2)$$

where $I_i := x_0 + G_i$ for each $i \in \mathbb{Z}$.

Now for $i < n$,

$$(\check{\psi}_j)^* b(x) = \int_G (\check{\psi}_j)^*(y) b(x-y) d\mu(y)$$

$$= \int_{I_{i+1}} + \int_{I_i \setminus I_{i+1}} + \int_{G \setminus I_i}$$

If $x \in I_i \setminus I_{i+1}$ and $y \in I_{i+1}$, $x-y \in G_i \setminus G_{i+1} \subset G \setminus G_n$. Also

if $x \in I_i \setminus I_{i+1}$ and $y \notin I_i$, $x-y \in G \setminus G_i \subset G \setminus G_n$. These, together with $\text{supp } b \subset G_n$ imply that the first and last terms of the right hand side of the equality are zero. Thus (4.2) is bounded by

$$\begin{aligned} &\sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{n-1} \int_{J_i} |((\check{\psi}_j)^* \xi_{J_i}) * b| d\mu_\alpha \\ &< \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{n-1} \int_G |b(y)| d\mu(y) \int_{J_i} |(\check{\psi}_j) \xi_{J_i}(x-y)| d\mu_\alpha(x) \end{aligned} \quad (4.3)$$

where $J_i := I_i \setminus I_{i+1}$ for each $i \in \mathbb{Z}$. Now, Lemma (c)

$$\begin{aligned} &\int_{J_i} |((\check{\psi}_j)^* \xi_{J_i})(x-y)| d\mu_\alpha(x) \\ &< \left(\int_{J_i} |((\check{\psi}_j)^* \xi_{J_i})(x-y)|^r d\mu(x) \right)^{1/r} \left(\int_{I_i} v_{\alpha r'}(x) d\mu(x) \right)^{1/r'} \\ &< \left(\int_{J_i} |((\check{\psi}_j)^*(x))|^r d\mu(x) \right)^{1/r} (m_i)^{-1/r'} \inf\{v_\alpha(x); x \in I_i, x \neq 0\} \\ &= \left(\int_{G \setminus G_{i+1}} |(\phi_j)^*(x)|^r d\mu(x) \right)^{1/r} (m_i)^{-1/r'} \inf\{v_\alpha(x); x \in I_i, x \neq 0\} \end{aligned}$$

Hence (4.3) is bounded by

$$\leq \sum_{j=n}^{\infty} \sum_{i=-\infty}^{n-1} \int_G |a(y+x_0)| d\mu(y) \inf\{v_\alpha(x); x \in I_i, x \neq 0\} \\ \times (m_i)^{-1/r'} \left(\int_{G_i \setminus G_{i+1}} |(\phi_j)^\vee(x)|^r d\mu(x) \right)^{1/r}$$

Since $I = I_n \subset I_1$ ($i < n$) and $\|a\|_{1,\alpha} < 1$,

$$B \leq C \sum_{j=11}^{\infty} \sum_{i=-\infty}^{n-1} (m_i)^{-1/r'} \left(\int_I |a(y)| d\mu(y) \inf\{v_\alpha(x); x \in I, x \neq 0\} \right. \\ \left. \times \left(\int_{G_i \setminus G_{i+1}} |(\phi_j)^\vee(x)|^r d\mu(x) \right)^{1/r} \right. \\ \leq C \sum_n^{\infty} \sum_{-\infty}^{n-1} m_i^\varepsilon (m_i)^{-(1/r' + \varepsilon)} \left(\int_{G_i \setminus G_{i+1}} |(\phi_j)^\vee|^r d\mu \right)^{1/r} \\ \leq C \sum_n^{\infty} \sum_{-\infty}^{n-1} m_i^\varepsilon \left\| |(\phi_j)^\vee| \right\|_{K_r^\varepsilon + 1/r', \infty} \leq C \sum_n^{\infty} (m_j)^{-\varepsilon} \sum_{-\infty}^{n-1} m_i^\varepsilon \leq C. \quad (4.4)$$

Hence, we have that $\|f\|_{1,\alpha}^* \leq C$ by (4.1) and (4.4). This completes the proof.

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