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BOUNDED SETS IN $\mathcal{L}$(E,F)

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Abstract: Let E and F be Hausdorff locally convex spaces, and let $\mathcal{L}(E,F)$ denote the space of continuous linear maps from E to F. Suppose that for every subspace $N \subset E$ and an absolutely convex set $A \subset E$ which is bounded, closed, and absorbing in N, there is a barrel $D \subset E$ such that $A = D \cap N$. Then it is shown that the families of weakly and strongly bounded subsets of $\mathcal{L}(E,F)$ are identical if and only if E is locally barreled.

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I. INTRODUCTION.

Throughout this paper E and F will denote Hausdorff locally convex spaces, and $\mathcal{L}(E,F)$ the space of continuous linear maps from E to F. An absolutely convex set B in E will be called a *disk*. If A is any subset of E, its linear hull will be denoted by E_A. For a disk B in E, its linear hull is given by $E_B = \cup\{nB : n \geq 1\}$. Equipped with the topology generated by the Minkowski functional of B, E_B is a semi-normed space. This leads to the definition which follows.

DEFINITION 1: Let $B \subset E$ be a disk. If E_B is a barreled normed space, then B is called a *barreled disk*; E is *locally barreled* if each bounded set in E is contained in a closed, bounded barreled disk.

II. A UNIFORM BOUNDEDNESS THEOREM

It is proven in [1] that in a locally convex space E the families of $\sigma(E', E)$-bounded and $\beta(E', E)$-bounded sets are the same if E is locally barreled. This is proven for the general case $\mathcal{L}(E, F)$ in our first result below. Conversely, in section III. we will examine the local barreledness of E in terms of subsets of $\mathcal{L}(E, F)$ which are bounded for any $\mathcal{S}$-topology, where $\mathcal{S}$ is a family of bounded sets which covers E.

THEOREM 2: If E is locally barreled then the families of bounded sets in $\mathcal{L}(E, F)$ are the same for all $\mathcal{S}$-topologies, where $\mathcal{S}$ is a family of bounded sets in E which covers E.

PROOF: Assume E to be locally barreled. Let V be a closed, absolutely convex 0-neighborhood in F. Let $H \subset \mathcal{L}(E, F)$ be pointwise bounded. Let

$$D = \bigcap \{u^{-1}(V) : u \in H\}.$$

Then D is a closed disk in E. Since H is pointwise bounded, we have:

$$x \in E \Rightarrow \bigcup \{u(x) : u \in H\} \subset \alpha V,$$

for some $\alpha > 0$. By taking inverse images, it follows that D is absorbing in E; hence, D is a barrel in E. In 8.5, Chapter II of [2] it is proven that D absorbs all bounded Banach disks. A careful reading of that proof reveals that the only property of Banach spaces which is used is the property of being barreled. Hence, any barrel in E absorbs all closed, bounded barreled disks in E, as well. Moreover, if A is any bounded subset of E, then A is contained in some closed, bounded barreled disk B. Therefore, D absorbs A and 3.3, Chapter III of [2] now asserts that H is bounded for the topology of bounded convergence on $\mathcal{L}(E, F)$. □

III. LOCALLY BARRELED SPACES AND BOUNDED SETS IN $\mathcal{L}(E, F)$.

Let (P) denote the following property of a locally convex space E:

(P) For each absolutely convex, closed, bounded set $A \subset E$ there exists a barrel $D \subset E$ such that $A = D \cap E_A$.

THEOREM 3: Let E and F be a Hausdorff locally convex spaces. Assume E satisfies property (P). Then the following are equivalent:

(a) The families of bounded subsets of $\mathcal{L}(E,F)$ are identical for all S-topologies on $\mathcal{L}(E,F)$, where S is a family of bounded subsets of E which covers E.

(b) E is locally barreled.

PROOF. In view of Theorem 2, we need only prove $(a) \Rightarrow (b)$.
If E is not locally barreled, then there exists an absolutely convex, closed, bounded set $B \subset E$ such that E_B is not barreled. We will first show that every set M which is closed and bounded in E_B is also closed in E. Denote by M_0 the closure of M in E. Since M is bounded in E_B, $M \subset \lambda B$, for some $\lambda > 0$. λB is closed in E. Hence $M_0 \subset \lambda B \subset E_B$. Take x_0 in M_0 and a net $\eta \subset M$ such that $\eta \longrightarrow x_0$ in the topology of E. The identity $id : E_B \longrightarrow E$ is continuous, and $\{k^{-1}B : k \in \mathbf{N}\}$ is a basis for the neighborhoods of zero in E_B consisting of sets closed in E. Therefore, by 3.2.4 of [3], $\eta \longrightarrow x_0$ in the topology of E_B. Finally, M is closed in E_B. Hence $x_0 \in M$, so M is closed in E.

Now choose a barrel A in E_B which is not a 0-neighborhood in E_B. Then we may choose a sequence $\{x_n\} \subset E_B \backslash A$ such that $x_n \longrightarrow 0$ in the topology of E_B. The normability of E_B implies that (x_n) is locally convergent; thus we may choose a sequence $\{a_n\}$ of positive real numbers such that $a_n \uparrow \infty$ and $a_n x_n \longrightarrow 0$ in the normed space E_B. Since the normed topology of E_B is finer than the topology on E_B induced by E, the sequence $\{a_n x_n\}$ also converges to 0 with respect to the topology of E. This means

$$S = \{a_n x_n : n \in \mathbf{N}\}$$

is bounded in E.

Since $A \cap B$ is absolutely convex, bounded, and closed in E_B, it is also closed and bounded in E. By (P), there is a barrel $D \subset E$ such that

$$A \cap B = D \cap E_{A \cap B} = D \cap E_B .$$

Now, $x_n \notin D$ for each n, and we may therefore choose $f_n \in E'$ such that $|f_n(x)| \leq 1$ for any $x \in D$ while $f_n(x_n) = 1$, where each f_n is real valued.

Let $y_0 \in F \backslash \{0\}$, and define $g : \mathbf{R} \longrightarrow F$ by

$$g(z) = z y_0,$$

for each $z \in \mathbf{R}$. g is a linear map taking bounded sets in $\mathbf{R}$ to bounded sets in F; therefore, g is continuous.

Now, for each $n \in \mathbf{N}$, define $h_n : E \longrightarrow F$ by

$$h_n = g \circ f_n.$$

As the composition of two linear, continuous maps, each $h_n \in \mathcal{L}(E, F)$.

Put

$$H = \{h_n : n \in \mathbf{N}\}.$$

First, notice that for each $x \in D$, $|f_n(x)| \leq 1$, hence $h_n(x) \in C$, where C is the line segment from $-y_0$ to y_0 in F. Obviously, C is bounded in F; consequently,

$$\bigcup\{h_n(x) : n \in \mathbf{N}\}$$

is bounded in F for each $x \in D$. Since D is absorbing in E,

$$\bigcup\{h_n(x) : n \in \mathbf{N}\}$$

is bounded in F for each $x \in E$ as well; this makes H a pointwise bounded set.

Finally,

$$\bigcup\{h_n(x) : x \in S, n \in \mathbf{N}\} = \bigcup\{h_n(a_n x_n) : n \in \mathbf{N}\} = \bigcup\{a_n g(1) : n \in \mathbf{N}\} = \bigcup\{a_n\{y_0\} : n \in \mathbf{N}\}.$$

Letting $\alpha_n = a_n^{-1}$, then

$$\lim_{n \to \infty} \alpha_n = 0,$$

while

$$\lim_{n \to \infty} \alpha_n(a_n\{y_0\}) = y_0 \neq 0.$$

This means $H(S)$ is not bounded in F; thus H is not bounded for the topology of uniform convergence on bounded sets. □

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