ON SOME FIXED POINT THEOREMS

D. ROUX and S.P. SINGH

Dipartimento di Matematica
Universite di Milano
Via C. Saldini 50
20133 Milano, Italy

Department of Mathematics
Memorial University
St. John's, NF, Canada, A1C 5S7

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ABSTRACT. In this paper we prove a fixed point theorem for inward mappings using a well-known result of Ky Fan type in Hilbert space setting.

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The following well known theorem of Ky Fan has been of great importance in nonlinear analysis, minimax theory and approximation theory [1].

Let C be a nonempty compact, convex subset of a normed linear space X and let $f: C \rightarrow X$ be a continuous mapping. Then there exists a $y \in C$ such that

$$\|y - fy\| = d(fy, C) ,$$

where $d(a, B) = \inf\{\|a - b\|/b \in B\}$.

If $fy \in C$, then f has a fixed point.

There have appeared several extensions of Ky Fan theorem. Lin [2] proved an interesting result for densifying mappings. Reich [3] relaxed compactness and proved the result for approximately compact, convex sets. Other results are due to Sehgal [4], Sehgal and Singh [5], Kapoor [6] and Singh and Watson [7].

In the present paper we prove a fixed point theorem for inward mappings using a result of Ky Fan type theorem for Hilbert space.

For definitions and notations we refer to Browder [8]. We will use his results for our theorem.

Let C be a closed, bounded, convex subset of H , a Hilbert space. A function $f: C \rightarrow H$ is called semicontractive if there exists a mapping T of $H \times H \rightarrow C$ such that

- i) $f(x) = T(x, x)$ for $x \in C$, while
- ii) for fixed $x \in H$, $T(\cdot, x)$ is nonexpansive,
- iii) for fixed $x \in H$, $T(x, \cdot)$ is compact.

Recall that $f: H \rightarrow H$ is nonexpansive if $\|fx - fy\| \leq \|x - y\|$ for all $x, y \in H$.

The following is a special case of a well-known theorem of Browder [8]. (We state it in Hilbert space).

Let C be a closed, bounded, convex subset of a Hilbert space H and let $f: C \rightarrow C$ be a semicontractive mapping. Then f has a fixed point.

The following more general result holds.

THEOREM 1. Let C be a nonempty, closed, convex subset of a Hilbert space H and let $f: C \rightarrow H$ be semicontractive mapping such that $f(C)$ is bounded. Then there exists a $y \in C$ such that

$$\|y - fy\| = d(fy, C).$$

PROOF: Let $P: H \rightarrow C$ be the proximity map. Then P is a non-expansive map, i.e.

$$\|Px - Py\| \leq \|x - y\| \text{ for all } x, y \in H. \text{ (see [9]) .}$$

Also,

$$Pof: C \rightarrow C.$$

Let $B = \overline{C_0}(Pf(C))$, convex closure of $(Pf(C))$.

Then $Pf: B \rightarrow B$ is a semicontractive mapping and has a fixed point say $Pfy = y$.

$$\text{Therefore } \|y - fy\| = \|Pfy - fy\|$$

$$= d(fy, C).$$

COROLLARY 1.

Let C be a closed, bounded and convex subset of H and let $f: C \rightarrow H$ be a semicontractive. Then there exists a $y \in C$ such that

$$\|y - fy\| = d(fy, C).$$

Let us now recall the "inwardness condition". Let K be a closed subset of a Banach space X . We say that $f: K \rightarrow X$ is an inward mapping if for every $x \in K$

$$f x \in I_K(x) = \{z: z = x + \alpha(y - x) \in K, \alpha \geq 0\}$$

This condition introduced by Halpern [10] and [11] is weaker than $x \in \delta K \Rightarrow f(x) \in K$ and is widely used in order to obtain fixed point results for mappings $f: K \rightarrow X$. See e.g. Assad and Kirk [12], Caristi [13], Caristi and Kirk [14], Downing and Kirk [15], S. Reich [3], Downing and Ray [16] and S. Massa [17], [18]. (δK stands for boundary of K).

S. Massa [18] pointed out that if K is a convex set C ($K=C$) then the inwardness condition is equivalent to

$$x \in C \Rightarrow (x, fx) \cap C \neq \emptyset$$

where $(x, y) = \{(1 - \alpha)x + \alpha y, 0 < \alpha \leq 1\}$.

THEOREM 2. Let C be a closed, convex subset of a Hilbert space H and $f: C \rightarrow H$ be a semicontractive inward mapping with bounded range. Then f has a fixed point.

PROOF. Let $y \in C$ be such that

$$\|y - fy\| = d(fy, C). \quad (\text{By Theorem 1}).$$

Suppose $y \neq fy$. Then $fy \notin C$ and there exists a $z \in (y, fy) \cap C$. We have

$$\|y - fy\| = \|y - z\| + \|z - fy\|.$$

Then $d(fy, C) \geq \|y - z\| + d(fy, C)$ absurd, because $y \neq z$.

COROLLARY 2.

Let C be a closed, convex subset of H and let $f: C \rightarrow H$ be semicontractive with bounded range. If $f(\delta C) \subseteq C$, then f has a fixed point.

COROLLARY 3.

Let B_r be a closed ball of radius r and center 0 in a Hilbert space H . Let $f: B_r \rightarrow H$ be a semicontractive mapping satisfying the condition: if $fx = \alpha x$ for $x \in \delta B_r$ then $\alpha \leq 1$. Then f has a fixed point.

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