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CONFORMAL VECTOR FIELDS IN SYMMETRIC AND CONFORMAL SYMMETRIC SPACES

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ABSTRACT. Consequences of the existence of conformal vector fields in (locally) symmetric and conformal symmetric spaces, have been obtained. An attempt has been made for a physical interpretation of the consequences in the framework of general relativity.

KEY WORDS AND PHRASES: Symmetric spaces, Conformal symmetric spaces, Conformal vector field, Null Killing vector field, Killing horizon.
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1. INTRODUCTION.

Let M denote a semi-Riemannian manifold with metric tensor g_{ab} of arbitrary signature. All the geometric objects defined on M are assumed sufficiently smooth. Although our treatment is local, nevertheless we shall drop the term `locally', for example in `locally symmetric'. We denote the Christoffel symbols by Γ^a_{bc} and the covariant differentiation by a semi-colon ; . We say that M is symmetric in Cartan's sense if the Riemann curvature tensor R^a_{bcd} is covariant constant, i.e. $R^a_{bcd;e} = 0$. We say that M is conformal symmetric [1] if its Weyl conformal curvature tensor $C^a{}_{bcd}$ is covariant constant, i.e. $C^a_{bcd;e} = 0$. Thus a symmetric space is conformal symmetric but the converse is not necessarily true.

A vector field ξ on M is said to be conformal if

$$L_\xi g_{ab} = 2\sigma g_{ab} \tag{1.1}$$

where L_ξ denotes the Lie-derivative operator via ξ and σ denotes a scalar function on M. In particular, if $\sigma = 0$, ξ is called a Killing vector field and if σ is a non-zero constant, ξ is called a homothetic vector field. It is known that a conformal vector field ξ satisfies:

$$L_\xi C^a{}_{bcd} = 0 \tag{1.2}$$

Equation (1.1) implies

$$L_\xi \Gamma^c_{ab} = \delta^c_a \sigma_{;b} + \delta^c_b \sigma_{;a} - g_{ab} g^{cd} \sigma_{;d} \tag{1.3}$$

but the converse is not necessarily true. However, we know [2] that (1.3) is equivalent to

$$L_\xi g_{ab} = 2\sigma g_{ab} + h_{ab} \tag{1.4}$$

where h_{ab} is a covariant constant tensor field. A vector field ξ satisfying (1.3) or (1.4) is said to be affine conformal [2] and is said to generate a one-parameter group of conformal collineations [3,4]. An affine conformal vector field with constant σ (i.e. $L_\xi \Gamma^a_{bc} = 0$) is known as an affine Killing vector field (which preserves the geodesics). For (i) a compact orientable positive definite Riemannian manifold without boundary, (ii) an irreducible positive definite Riemannian manifold and (iii) an $n(n > 2)$ - dimensional non-flat space-form; an affine conformal vector field reduces to conformal vector field. For a non-Einstein conformally flat space of dimension > 2; Levine and Katzin [5] proved that h_{ab} is a linear combination of g_{ab} and the Ricci tensor R_{ab}.

Conformal motion (generated by a conformal vector field) is a natural symmetry of the space-time manifolds in general relativity, inherited by its causality-preserving [6] character. But sometimes, it is desirable to consider conformal motions which provide covariant conservation law generators. It was pointed out by Katzin et al [7] that there is a fundamental symmetry called curvature collineation (CC) defined by a vector field ξ satisfying

$$L_\xi R^a{}_{bcd} = 0 \tag{1.5}$$

Komar's identities [8] (which define a conservation law generator) follow naturally by the existence of CC. A conformal vector which also generate CC, is called a special conformal vector. A conformal vector is special conformal vector iff

$$\sigma_{;ab} = 0 \tag{1.6}$$

The purpose of this paper is to study the consequences of the existence of (i) a special conformal vector field in a symmetric space and (ii) a conformal vector field in a conformal symmetric space; and indicate the physical interpretation of the consequences within the framework of general relativity.

2. SYMMETRIC AND CONFORMAL SYMMETRIC SPACES.

Here we prove two theorems as follows:

THEOREM 1. *Let a non-flat symmetric space* M *of dimension* $n > 4$, *admit a special conformal vector field* ξ. *Then either* (i) M *has zero scalar curvature and grad* σ *is a null Killing vector field, or* (ii) ξ *reduces to a homothetic vector field.*

(Note that the above theorem is valid also for an affine conformal vector field, in which case the alternative conclusion (ii) would be: ξ reduces to an affine Killing vector field. The proof is common).

PROOF. We have the following identity [9]:

$$L_\xi(R^b{}_{cde;a}) - (L_\xi R^b{}_{cde})_{;a} = (L_\xi \Gamma^b_{af})R^f{}_{cde} - (L_\xi \Gamma^f_{ac})R^b{}_{fde} - (L_\xi \Gamma^f_{ad})R^b{}_{cfe} - (L_\xi \Gamma^f_{ae})R^b{}_{cdf}$$

By our hypothesis the left hand side vanishes and consequently, in view of (1.3) the above equation assumes the form:

$$\begin{aligned}\delta^b_a \sigma_f R^f{}_{cde} + \sigma_a R^b{}_{cde} - \sigma^b R_{acde} &= \sigma_c R^b{}_{ade} \\ &+ \sigma_a R^b{}_{cde} - g_{ac} R^b{}_{fde} \sigma^f + \sigma_d R^b{}_{cae} + \sigma_a R^b{}_{cde} \\ &- g_{ad} R^b{}_{cfe} \sigma^f + \sigma_e R^b{}_{cda} + \sigma_a R^b{}_{cde} - g_{ae} R^b{}_{cdf} \sigma^f\end{aligned} \tag{2.1}$$

where σ_a stands for $\sigma_{;a}$. Taking the product of both sides with σ^a yields

$$(\sigma_a \sigma^a) R^b{}_{cde} = 0$$

As per our hypothesis, M is not flat and therefore the above equation shows

$$\sigma_a \sigma^a = 0$$

which implies that either (i) grad σ is null (a non-zero vector of zero norm), or (ii) σ is constant. We first take up case (i). Successive contractions of (2.1) lead to

$$(n - 4)\sigma_f R^f{}_e = R \sigma_e \tag{2.2}$$

Two subcases arise: If $n = 4$, then (2.2) gives $R = 0$. If $n > 4$, then using the condition (1.6) obtains $R^f{}_e \sigma_f = 0$. This, substituted in (2.2), shows that $R = 0$. Thus, in case (i) grad σ is null and Killing (in virtue of (1.6)) and the scalar curvature R vanishes identically. In case (ii) ξ is homothetic or affine Killing according as ξ is conformal or affine conformal. This proves the theorem.

THEOREM 2. *Let a conformal symmetric space* M (dim M > 3) *admit a conformal vector field* ξ. *Then one of the following holds:*

(i) M *is conformally flat*

(ii) grad σ *is a null vector*

(iii) ξ *reduces to a homothetic vector field.*

In particular, if ξ *were non-homothetic special conformal vector field and* M *were not conformally flat, then* grad σ *would have been null and Killing too.*

PROOF. Consider the identity [9]:

$$\begin{aligned}L_\xi(C^b{}_{cde;a}) - (L_\xi C^b{}_{cde})_{;a} &= (L_\xi \Gamma^b_{af})C^f{}_{cde} - (L_\xi \Gamma^f_{ac})C^b{}_{fde} \\ &- (L_\xi \Gamma^f_{ad})C^b{}_{cfe} - (L_\xi \Gamma^f_{ae})C^b{}_{cdf}\end{aligned}$$

Observe that the left hand side vanishes because M is conformal symmetric and (1.2) holds for a conformal vector field. By use of (1.3) in the above equation and contracting at a and b we obtain (noting dim M > 3)

$$\sigma^a \sigma_a C^b{}_{cde} = 0$$

Therefore we conclude that either (i) $C^b{}_{cde} = 0$ meaning M is conformally flat, or $\sigma^a \sigma_a = 0$ so that (ii) σ^a is a null vector or (iii) σ is constant. Thus we have proved that one of the following is true: (i) M is conformally flat, (ii) grad σ is a null vector, (iii) ξ is homothetic. In particular, if ξ were non-homothetic special conformal vector field and M not conformally flat then, of course, (ii) holds. Moreover, in this case grad σ would be Killing in virtue of the condition (1.6) for special conformal vector field. This completes the proof.

REMARK 1. The conclusion (i) of Theorem 2 can be highlighted by saying that, if a conformal symmetric space M admits a one-parameter group of conformal motions (such that grad σ is neither null nor zero) then M is conformally flat. This can be compared with the standard result:[9] If an n-dimensional semi-Riemannian manifold M admits a maximal, i.e. $\frac{1}{2}(n+1)(n+2)$ - parameter group of conformal motions, then M is conformally flat.

REMARK 2. The conclusion (i) of Theorem 1 can be interpreted in the context of general relativity as follows. Let M be the space-time manifold of general relativity and satisfy the hypothesis of Theorem 1. M with zero scalar curvature, is a space-time carrying pure radiation [10] (e.g. massless scalar fields, neutrino fields or high frequency gravitational waves) and Einstein-Maxwell field. M with the gradient of conformal scalar field as a null Killing field, has a Killing horizon [11] defined by the null hypersurfaces of transitivity, σ = constant.

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REFERENCES

1. McLENAGHAN, R.G. and LEROY, J., Complex recurrent space-times, Proc. Roy. Soc. Lond. A 327 (1972), 229-249.

2. SHARMA, R. and DUGGAL, K.L., Characterization of an affine conformal vector field, C.R. Acad. Sci. Canada, 7 (1985), 201-205.

3. TASHIRO, Y., On conformal collineations, Math. J. Okayama Univ., 10 (1960), 75-85.

4. DUGGAL, K.L. and SHARMA, R., Conformal collineations and anisotropic fluids in general relativity, J. Math. Phys. 27 (1986), 2511-2513.

5. LEVINE, J. and KATZIN, G.H., Conformally flat spaces admitting special quadratic first integrals, I. Symmetric spaces, Tensor N.S. 19 (1968), 317-328.

6. HAWKING, S.W. and ELLIS, G.F.R., The Large Scale Structure of Space-time (Cambridge University Press, Cambridge), 1973.

7. KATZIN, G.H., LEVINE, J. and DAVIS, W.R., Curvature collineations: A fundamental symmetry property of the space-times of general relativity defined by the vanishing Lie-derivative of the Riemann curvature tensor, J. Math. Phys., 10 (1969), 617-629.

8. KOMAR, A., Covariant conservation laws in general relativity, Phys. Rev. 113 (1959), 934-936.

9. YANO, K., Integral formulas in Riemannian geometry, Marcel Dekker, N.Y., 1970.

10. KRAMER, D., STEPHANI, H., HERLT, E. and MacCALLUM, M., Exact Solutions of Einstein's field equations, Cambridge University Press, Cambridge, 1980.

11. CARTER, B., Killing horizons and orthogonally transitive groups in space-time, J. Math. Phys., 10 (1969), 70-81.