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ON THE NUMBER OF CUT-VERTICES IN A GRAPH

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ABSTRACT. A connected graph with n vertices contains no more than $\frac{r}{2r-2}$ (n-2) cut-vertices of degree r. All graphs in which the bound is achieved are described. In addition, for graphs of maximum degree three and minimum δ, best possible bounds are obtained for $\delta = 1, 2, 3$.

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1. INTRODUCTION.

It is well known that a connected graph with n vertices, $n \geq 2$, has at least 2 non-cut-vertices, and hence, at most n-2 cut vertices. The only graphs with n-2 cut-vertices are paths, and, in these, every cut-vertex has degree 2. The purpose of this article is to show that the above is a special case of a more general result, namely that in any graph the number of cut-vertices of degree r is less than or equal to $\frac{r}{2r-2}$ (n-2). We prove the case $r \geq 3$ of this result, the case r=2 being known, as mentioned above. In addition, we demonstrate all graphs in which the upper bound is achieved. Similar questions were considered in [1] and [2], where the principal results involve regular graphs. We begin with some definitions.

Let M be a vertex set in the connected graph G. An <u>M-block</u> is a set B of vertices maximal with respect to the following property: if $x,y \in B$, there exists no vertex v in M distinct from x and y such that the removal of v from G leaves x and y in different components. A cut vertex which lies in M is called an <u>M-cut-vertex</u>. In strict analogy with the standard block-cutpoint-tree construction [3], we define the <u>M-block-cutpoint-tree</u> of a connected graph G to be the graph $T_M(G)$ whose vertices are the M-blocks and M-cut-vertices of G, with an M-cut-vertex v adjacent to an M-block B if and only if $v \in B$. It is easy to show that $T_M(G)$ is a tree. In case M is the set of all vertices of degree r in G, we refer to <u>r-cut-vertices</u>. The number of r-cut-vertices in the graph G will be denoted $\underline{c_r(G)}$ or simply $\underline{c_r}$. In case M is the set of all vertices of G, our notions coincide with the usual notions of block, cut-vertex, and block-cutpoint tree. If G is a graph, we will denote by S(G) the graph formed by joining to each vertex of G a new vertex of degree one, or end vertex. As usual, the complete graph with n vertices will be denoted K_n. An r-tree is a tree in which every vertex has degree r or degree 1.

Our proof that $c_r \leq \frac{r}{2r-2}(n-2)$ will be by induction on n. We begin with the following simple observation.

LEMMA 1. Let G be a graph with n vertices. Suppose that $c_r(G) > \frac{r}{2r-2}(n-2)$ and that n is the smallest number of vertices for which this is the case. Then no vertex of G is adjacent to two or more end vertices.

PROOF. Removal of two end vertices adjacent to a single vertex would decrease c_r by at most one. In the subgraph H which remains, we would have $\frac{c_r(H)}{(n-2)-2} \geq \frac{c_r-1}{n-4} > \frac{c_r}{n-2}$, contradicting the minimality of n.

LEMMA 2. Let r be an integer, $r \geq 3$. Let G be a connected graph with n vertices, $1 \leq n \leq 2r$. If $\frac{c_r(G)}{n-2} \geq \frac{r}{2r-2}$, then n=2r and G = $S(K_r)$, which implies that $\frac{c_r}{n-2} = \frac{r}{2r-2}$.

PROOF. Suppose G is a smallest graph in which the inequality holds. Since it holds in $S(K_r)$, we know $n \leq 2r$. Let v be an r-cut-vertex in G. Then no more than one component of G - v may contain vertices of degree r (degree taken in G), or else G would have at least 2r+1 vertices. Suppose that component C of G - v contains all vertices of degree r in G - v. Then the remainder of G - v consists of a single vertex. For, if there are two or more vertices in G - v - C, then we remove them and argue as in Lemma 1. We now know that each r-cut-vertex in G is adjacent to an end vertex. Hence, G consists of c_r r-cut-vertices, c_r end vertices adjacent to these, and, let us say, j other vertices. So $n = 2c_r+j$ and our hypothesis says $\frac{c_r}{2c_r+j-2} \geq \frac{r}{2r-2}$. This implies that $2c_r + r(j-2) \leq 0$, which forces j = 0 or j = 1.

If j = 1, we have $r \geq 2c_r$ and $n = 2c_r + 1$, so $n \leq r + 1$. But any r-cut-vertex, along with its neighbors, accounts for r + 1 vertices, so n = r + 1, and any vertex of degree r is adjacent to every vertex in G. Thus $c_r = 1$, r = 2, and we may rule out this case since we have assumed $r \geq 3$.

The remaining case is j = 0. Here we have $r \geq c_r$ (from $2c_r + r(j-2) \leq 0$), so $2r \geq 2c_r + j = n$. Since $n \leq 2r$, we have n = 2r and G = $S(K_r)$.

2. MAIN RESULTS.

THEOREM 1. Let $r \geq 3$ be an integer. Let G be a connected graph with n vertices. Then $c_r(G) \leq \frac{r}{2r-2}(n-2)$. In case equality occurs, G is the graph formed from some r-tree by replacing each vertex of degree r by K_r.

PROOF. We argue by induction on n. Lemma 2 settles the case $n \leq 2r$, since S(Kr) is formed from an r-star by the construction mentioned. Suppose now that $n \geq 2r$ and that the statement holds for graphs with fewer vertices. If every r-cut-vertex in G is adjacent to an end vertex, then certainly $c_r \leq \frac{1}{2}n$ and it follows that

$\frac{c_r}{n-2} < \frac{r}{2r-2}$. Now suppose that there are r-cut-vertices adjacent to no end vertex. Let M be the set of all such, and let B be an end block in the M-block-cutpoint tree of G. Let v be the unique M-cut-vertex in B. Let H be the subgraph of G induced by v and the vertices not in B. By induction $c_r(H) \leq \frac{r}{2r-2}(h-2)$ where h is the number of vertices in H. Let j be the number of r-cut-vertices other than v in B. Note that each of these j vertices is adjacent to exactly one end vertex, by Lemma 1. Let ℓ be the number of vertices in B other than the r-cut-vertices and the end vertices. Then $|B| = 2j + \ell + 1$. Now, we have $\frac{c_r(G)}{n-2} = \frac{c_r(H) + 1 + j}{n-2} = \frac{c_r(H) + 1 + j}{h + |B| - 1 - 2} = \frac{c_r(H) + 1 + j}{h + 2j + \ell - 2} \leq \frac{\frac{r}{2r-2}(h-2) + 1 + j}{h + 2j + \ell - 2}$. Simple arithmetic shows this ratio is less than or equal to $\frac{r}{2r-2}$ if and only if $2(j+1) + (\ell-2)r \geq 0$. This is clear if $\ell \geq 2$. If $\ell = 1$, we must show that $r \leq 2j + 2$. Let u be the vertex in B which is neither an end vertex nor an r-cut-vertex. If $j \neq 0$, let w be one of the j r-cut-vertices adjacent to an end vertex. Then w has exactly r neighbors. These are included among u, v, the end vertex adjacent to w, and the j-1 remaining r-cut-vertices in B. Hence, $r \leq j + 2 < 2j + 2$ and we have strict inequality in this case. If j=0, then B consists only of v and u, which contradicts the fact that v is adjacent to no end vertex. Now suppose $\ell = 0$. We must show $r \leq j + 1$. The case $j = 0$ is not possible, since this would leave B with a single vertex. So, let w be one of the j r-cut-vertices adjacent to an end vertex. The possible neighbors of w are its end vertex neighbor, v, and the remaining $j - 1$ r-cut-vertices. Hence, $r \leq j + 1$.

Now, if equality $\frac{c_r}{n-2} = \frac{r}{2r-2}$ holds in G, we must be in the case $\ell = 0$, and we must have $c_r(H) = \frac{r}{2r-2}(h-2)$, and $r = j + 1$. By induction, H is a graph formed by replacing each r-vertex in some r-tree by K_r. Since $r = j + 1$, the block B consists of v, r-1 mutually adjacent r-cut-vertices, and end vertices adjacent to each of these r-1. Hence, since v has degree r, it has only one neighbor outside B, and B is precisely a copy of $S(K_r)$ attached to H by a single edge. Hence, G is of the desired structure.

The bound $\frac{r}{2r-2}(n-2)$ is achieved only in graphs with many vertices of degree one. It would be interesting to know what upper bound might be achieved in graphs with larger minimum degree. We are able to answer this question rather easily for graphs of maximum degree three, using the fact that in such graphs every cut vertex is incident with a bridge.

THEOREM 2. Let G be a graph with maximum degree 3. Then:

(i) If G has no vertices of degree less than 2, then $c_3(G) \leq \frac{2}{3}(n-3)$;

(ii) If G is three-regular, then $c_3(G) \leq \frac{1}{2}(n-6)$.

PROOF. i) The proof is by induction on n. We may assume that no bridge is incident with a vertex of degree two, for, otherwise, eliminating (smoothing out) such a vertex would result in a graph with fewer vertices in which c_3 remains unchanged. Now, if G has no bridge, then $c_3 = 0 \leq \frac{2}{3}(n-3)$. If G has bridges, then remove a bridge joining vertices v_1 and v_2, both of degree 3. The two components

G_1 and G_2 with n_1 and n_2 vertices respectively, have minimun degree: two, and by induction, G_1 and G_2 have at most $\frac{2}{3}$ (n_1-3) and $\frac{2}{3}$ (n_2-3) cut vertices of degree three. Adding 2 to include v_1 and v_2, we have $c_3(G) \leqq 2 + \frac{2}{3}(n_1-3) + \frac{2}{3}(n_2-3) = \frac{2}{3}(n-3)$.

(ii) Again we proceed by induction. It suffices to prove the statement for cubic graphs in which no cut vertex is incident with three bridges, for, blowing up all such vertices to triangles would result in a graph all of whose extra vertices are cut vertices, and in which no cut vertex is incident with three bridges. Now, when a bridge is removed, each component must have at least five vertices. If there is a bridge each of whose components has more than five vertices, remove it, leaving components H_1 and H_2 with n_1 and n_2 vertices. H_i has a unique vertex v_i of degree two, $i = 1, 2$. Form now graphs G_1 and G_2 by joining copies of K_4 with a subdivided edge (which has also a unique vertex of degree two) to v_1 and v_2. Since n_1 and n_2 are bigger than 5, it follows that the graphs G_1 and G_2, which have orders n_1+5 and n_2+5 (that is $n-n_2+5$ and $n-n_1+5$) are smaller that G. By induction, G_1 and G_2 have at most $\frac{1}{2}(n_1+5-6)$ and $\frac{1}{2}(n_2+5-6)$ cut vertices. This counts two vertices wich are not cut vertices of G. Hence $c_3(G) \leqq \frac{1}{2}(n_1-1) + \frac{1}{2}(n_2-1) - 2 = \frac{1}{2}(n-6)$. All that remains is the case where the removal of any bridge leaves at least one component of exactly five vertices. In this case, removal of all bridges leaves several components of five vertices and at most one other component with, say K vertices. If this component has r vertices of degree 2, then G has $K + 5r$ vertices and $2r$ cut vertices. So, $\frac{c_3}{n} = \frac{2r}{K+5r} \leqq \frac{1}{3}$, so $c \leqq \frac{1}{3}n$. But $\frac{1}{3}n < \frac{1}{2}(n-6)$ for $n \geqq 18$. If $n < 18$, then $r \leqq 2$. If $r = 1$, then $c_3 = 2$ and $n \geqq 10$. If $r = 2$, then $c_3 = 4$, and it is easy to show that $n \geqq 14$. Hence, the inequality holds in every case.

Part (ii) of Theorem 4 is a special case of a result of [2]. The bounds of Theorem 4 are achieved in the following graphs. Let T be a tree in which every vertex has degree one or three. Replace each vertex of T by a triangle. The bound of (i) is achieved in this graph. To achieve the bound of (ii), replace each vertex in T of degree three by a triangle and each vertex of degree one by a subdivided K_4. These examples show that the results of Theorem 4 are best possible.

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