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NON ARCHIMEDEAN METRIC INDUCED FUZZY UNIFORM SPACES

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ABSTRACT. It is shown that the category of non-Archimedean metric spaces with 1-Lipschitz maps can be embedded as a coreflective non-bireflective subcategory in the category of fuzzy uniform spaces. Consequential characterizations of topological and uniform properties are derived.

KEYWORDS AND PHRASES. Non-Archimedean, coreflective, completion, fuzzy uniform space.
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1. INTRODUCTION.

We show that the category NA(1) of non-Archimedean metric spaces with metric bounded by 1 and with morphisms the non-expansive maps is coreflectively embedded in the category FUS of fuzzy uniform spaces [4], [9] in an extremely simple and natural way. Through the forgetful functor FUS → FNS [5] each space in NA(1) then moreover determines a non-topologically generated space in FNS, the topological modification (i.e. TOP-coreflection) of which is nothing else then the metric topology. This means that the diagram

$$\begin{tikzcd} \mathrm{NA}(1) \arrow[r, "\text{embedding}"] \arrow[d, "\text{forgetful functor}"] & \mathrm{FUS} \arrow[d, "\text{forgetful functor}"] \\ \mathrm{TOP} & \mathrm{FNS} \arrow[l, "\text{coreflection}"] \end{tikzcd}$$

is commutative. From a local point of view, an interesting aspect of this situation is that given (X,d) we can study this space first with the concepts available in FUS [4], [9] and with those available in FNS [5], [6] before going to the topological space

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$(X,\mathbf{T}_d)$ associated with (X,d). E.g. we can "forget" at intermediate stages. From a global point of view embedding NA(1) in FUS also seems natural. E.g. the functor NA(1) → TOP does not preserve products. NA(1), although being coreflectively embedded in FUS, is not bireflectively embedded, in particular the embedding does not preserve products, but it are precisely the products in FUS and in FNS which are mapped onto the topological product in TOP (TOP is both coreflectively and bireflectively embedded in FNS). Thus in order to have a more faithful relation with TOP it seems suitable to consider NA(1) as a subcategory of FUS. In particular we further study completeness of NA(1)-objects in FUS, and we also give a fairly complete account of the most important topological properties of NA(1)-objects in FNS.

2. PRELIMINARIES.

Most notions used are standard, we just recall some notations and some concepts specific to the context.

As always $\mathbb{R}_0^+$ stands for the strictly positive real numbers, $I := [0,1]$, $I_0 :=]0,1]$ and $I_1 := [0,1[$.

If X is a set and $A \subset X$, 1_A stands for the characteristic function of A. If $\lambda \in I^X$ and $\psi \in I^{X\times X}$ then $\psi\langle\lambda\rangle \in I^X$ is given by $\psi\langle\lambda\rangle(x) = \sup_{y\in X} \lambda(y) \wedge \psi(y,x)$. If $\lambda = 1_{\{x\}}$ we we simply put $\psi\langle x\rangle$ and obviously $\psi\langle x\rangle(y) = \psi(x,y)$. Also $\psi \circ \psi \in I^{X\times X}$ is given by $\psi \circ \psi(x,y) = \sup_{z\in X} \psi(x,z) \wedge \psi(z,y)$.

Further $\dot{\lambda}$ stands for the prefilter $\{\mu | \lambda \leq \mu\}$ and if $\mathbf{B} \subset I^X$ is a prefilter base then $\tilde{\mathbf{B}}$ stands for the prefilter $[\{\sup_{\varepsilon\in I_0} (\beta_\varepsilon - \varepsilon) | (\beta_\varepsilon)_\varepsilon \in \mathbf{B}^{I_0}\}]$.

If d is a pseudometric on X then we put $\mathbf{T}_d$ and $\mathbf{U}_d$ resp. the associated topology and uniformity.

If d fulfils the strong (or ultrametric) triangle inequality we call it a non-Archimedean pseudometric.

The functors ι, ι_α, ω, ω_u are well-known [3], [4], [5] but we recall the functors $\iota_{u,\alpha}$; FUS → UNIF determined by

$$\iota_{u,\alpha}(\mathbf{U}) := \{\mu^{-1}(]\beta,1]) | \beta \in [0,1-\alpha[, \mu \in \mathbf{U}\}$$

and t : FUS → FTS where then $t(\mathbf{U})$ stands for the fuzzy topology associated with $\mathbf{U}$ and T : UNIF → TOP where then $T(\mathbf{U})$ stands for the topology associated with $\mathbf{U}$. For a prefilter $\mathbf{F}$ [3], we recall also that its characteristic value is given by

$$c(\mathbf{F}) := \inf_{\lambda\in\mathbf{F}} \sup_{x\in X} \lambda(x) = \inf\{\alpha \in I | \alpha \in \mathbf{F}\}.$$

If $(X,\mathbf{U}) \in |\mathrm{FUS}|$ and $\mathbf{C}$ is a prefilter on X then it is called a hyper Cauchy prefilter [9] if it satisfies the conditions

(HC1) $c(\mathbf{C}) = 1$

(HC2) $\tilde{\mathbf{C}} = \mathbf{C}$

(HC3) $\forall\, \nu \in \mathbf{U}$, $\forall\, \varepsilon \in I_0$, $\exists\, \mu_\varepsilon \in \mathbf{C} : \mu_\varepsilon \times \mu_\varepsilon - \varepsilon \leq \nu$.

In [9] it was shown that for any hyper Cauchy prefilter $\mathbf{C}$, there exists a unique minimal hyper Cauchy prefilter $\mathbf{C}_0 \subset \mathbf{C}$. Moreover, if B is a basis for $\mathbf{C}$ and $\mathbf{W}$ a basis for $\mathbf{U}$

then

$$\mathbf{C}_0 = \{\nu\langle\mu\rangle \mid \nu \in \mathbf{V},\ \mu \in \mathbf{B}\}^{\sim}.$$

A fuzzy uniform space $(X,\mathbf{U})$ is called ultracomplete [9] if for each minimal hyper Cauchy prefilter $\mathbf{C}$ there exists $x \in X$ such that $\mathbf{C} = \mathbf{U}(x)$ where $\mathbf{U}(x) := \{\mu\langle x\rangle \mid \mu \in \mathbf{U}\}$, which is equivalent to the fact that $(X, \iota_u(\mathbf{U}))$ is complete.

Finally, $(X,\mathbf{U})$ is called precompact [9] if it satisfies the condition : $\forall\, x \in X,\ \forall\, \varepsilon \in I_0,\ \exists\, Y \in 2^{(X)} : \sup_{x\in Y} \nu\langle x\rangle \geq 1 - \varepsilon$, which is also equivalent to the fact that $(X, \iota_u(\mathbf{U}))$ is precompact.

3. DEFINITIONS AND FUNDAMENTAL PROPERTIES.

We first put together some elementary technical properties.

LEMMA 3.1.

1° If X is a set and $d \leq 1$ a non-Archimedean pseudometric on X, then $\psi_d := 1 - d : X \times X \to I$ has the following properties :

a. $\forall\, x \in X : \psi_d(x,x) = 1$;

b. ψ_d is symmetric;

c. $\psi_d \circ \psi_d = \psi_d$, or equivalently :

$$\forall\, (x,y,z) \in X^3 : \psi_d(x,z) \wedge \psi_d(z,y) \leq \psi_d(x,y).$$

2° If conversely $\psi \in I^{X\times X}$ has the properties

a. $\forall\, x \in X : \psi(x,x) = 1$

b. ψ is symmetric

c. $\psi \circ \psi = \psi$,

then $d_\psi := 1 - \psi$ is a non-Archimedean pseudometric on X for which $d_\psi \leq 1$.

3° If $d \leq 1$ is a non-Archimedean pseudometric on X, and if we put

$$D_r = \{(x,y) \mid d(x,y) < r\},$$

$$B(x,r) = \{y \mid d(x,y) < r\},$$

then

$$\psi_d^{-1}(]r,1]) = D_{1-r},\ (\psi_d\langle x\rangle)^{-1}(]r,1]) = B(x,1-r).$$

PROOF. Straightforward. ∎

In the sequel, if no confusion can arise, we simply put ψ resp. d instead of ψ_d resp. d_ψ.

THEOREM 3.2. If $d \leq 1$ is a non-Archimedean pseudometric on a set X, then $\{\psi\}$, with $\psi := 1 - d$, is a basis for a fuzzy uniformity $\mathbf{U}(d)$ on X, where

$$\mathbf{U}(d) := \{\psi\}^{\sim} = \dot{\psi}.$$

Conversely, if $\mathbf{U}$ is a fuzzy uniformity on X, having a singleton basis $\{\psi\}$, then this function ψ satisfies the conditions a, b, c in Lemma 3.1.2°, and therefore $\mathbf{U} = \mathbf{U}(d)$ where $d := 1 - \psi$ is a non-Archimedean pseudometric.

PROOF. The first part follows from Lemma 3.1.1° and the second part from the definition of a basis of a fuzzy uniformity [4] and an application of Lemma 3.1.2°. ∎

We now describe the general properties of $\mathbf{U}(d)$, where it is always supposed that d is a non-Archimedean pseudometric such that $d \leq 1$.

PROPOSITION 3.3. The following hold :

1° for all $\alpha \in I_1$, a basis for $\iota_{u,\alpha}(\mathbf{U}(d))$ is given by $\{D_r | r \in]\alpha,1]\}$, and therefore $\iota_{u,\alpha}(\mathbf{U}(d)) \subset \mathbf{U}_d$;

2° for all $(\alpha,x) \in I_1 \times X$, the neighborhoodfilter $\mathbf{N}_\alpha(x)$ of x in $\iota_\alpha(t(\mathbf{U}(d)))$ has $\{B(x,r) | r \in]\alpha,1]\}$ as a basis;

3° $\iota_u(\mathbf{U}(d)) = \mathbf{U}_d$ and so $T(\iota_u(\mathbf{U}(d))) = \mathbf{T}_d$;

4° for all $\alpha \in I_1$: $\iota_\alpha(t(\mathbf{U}(d))) = T(\iota_{u,\alpha}(\mathbf{U}(d)))$;

5° for all $\mu \in I^X$, the closure $\bar{\mu}$ and the interior $\mathring{\mu}$ of μ in $t(\mathbf{U}(d))$ are given by $\bar{\mu}(x) = \sup_{y\in X} \mu(y) \wedge \psi(y,x)$, $\mathring{\mu}(x) = \inf_{y\in X} \mu(y) \vee d(y,x)$.

6° $\lambda \in t(\mathbf{U}(d))$ iff there exists a partition P_λ of X by means of balls, i.e. a subset $Y \subset X$ and a function $\rho : Y \to I$ with $P_\lambda = \{B(x,\rho_x) | x \in Y\}$ such that $\lambda | B(x,\rho_x) = \rho_x$;

7° $\iota_\alpha(t(\mathbf{U}(d)))$ is Hausdorff iff $d(x,y) > \alpha$ for all $x \neq y$;

8° $(X,\mathbf{U}(d))$ is WT_2 iff d is metric;

9° $(X,\mathbf{U}(d))$ is T_2 iff $d(x,y) = 1$ for all $x \neq y$, i.e. iff d is the discrete metric.

PROOF. 1° Immediate from the definition of $\iota_{u,\alpha}(\mathbf{U}(d))$ and from $\psi^{-1}(]1-r,1]) = D_r$.
2° From Lemma 3.1.3°, and the fact that $(X,\mathbf{U}(d))$ is a fuzzy neighborhood space.
3° From 1° and $\iota_u(\mathbf{U}(d)) = \iota_{u,0}(\mathbf{U}(d))$.
4° This is a know property of general fuzzy uniform spaces.
5° Immediate from Proposition 2.4 in [5].
6° If $\lambda \in t(\mathbf{U}(d))$, $x \in X$, $\lambda(x) = \alpha$ and $d(x,y) < \alpha$, it follows from

$$\lambda(y) = \mathring{\lambda}(y) = \inf_{t\in X} \lambda(t) \vee d(t,y) \leq \lambda(x) \vee d(x,y)$$

that $\lambda(y) \leq \alpha$. However, we also have

$$\alpha = \lambda(x) = \mathring{\lambda}(x) = \inf_{t\in X} \lambda(t) \vee d(t,x) \leq \lambda(y) \vee d(x,y),$$

hence $\lambda(y) = \alpha$.
This means that $\lambda^{-1}(\alpha) = \cup\{B(x,\alpha) | \lambda(x) = \alpha\}$, and as each two of these balls are either identical or disjoint, we can choose $Y_\alpha \subset X$ such that $\{(x,\alpha) | x \in Y_\alpha\}$ is a disjoint family with $\lambda^{-1}(\alpha)$ as a union. Putting $Y := \bigcup_{\alpha\in I} Y_\alpha$, $\rho_x := \alpha$ iff $x \in Y_\alpha$, we are done.
Conversely, if $\lambda \in I^X$ is such that the described partition exists and if $\alpha \in I_1$, we have

$$\lambda^{-1}(]\alpha,1]) = \cup\{B(x,\rho_x) | x \in \rho^{-1}]\alpha,1]\},$$

which is clearly open in $\iota_\alpha(t(\mathbf{U}(d)))$. Since $(X,t(\mathbf{U}(d)))$ is a fuzzy neighborhood space, $t(\mathbf{U}(d))$ is maximal for its level topologies [13], and therefore $\lambda \in t(\mathbf{U}(d))$.

7° This follows from the fact that, by 1°, $\iota_\alpha(t(\mathbf{U}(d))$ is Hausdorff iff $\cap_{r\in]\alpha,1]} D_r$ is the diagonal of $X \times X$.

8° This is nothing else than the definition of WT_2.

9° $(X,\mathbf{U}(d))$ is T_2 iff $\psi(x,y) = 0$ for $x \neq y$, so iff $d(x,y) = 1$ for $x \neq y$. ∎

REMARKS 3.4. 1. If $d \leq 1$ and $d' \leq 1$ are equivalent, the fuzzy uniformities $\mathbf{U}(d)$ and $\mathbf{U}(d')$ are nevertheless in general different.

2. In the foregoing it was always supposed that $d \leq 1$. Starting from an arbitrary d, we can define a family of fuzzy uniformities. Indeed, given the non-Archimedean pseudometric d on X and $\varepsilon \in \mathbb{R}_0^+$, we can define $d_\varepsilon = (\frac{1}{\varepsilon} d) \wedge 1$, which is equivalent to d, and consider $\mathbf{U}(d_\varepsilon) := \dot{\psi}_\varepsilon$, where $\psi_\varepsilon := 1 - d_\varepsilon = (1-\frac{1}{\varepsilon} d) \vee 0$. Even in this case the fuzzy uniformities $\mathbf{U}(d_\varepsilon)$, $\varepsilon \in \mathbb{R}_0^+$, are in general not equivalent to each other, (some interesting relations will be established in Propositions 3.5 and 4.), e.g. if $X := E^{\mathbb{N}}$ where $|X| \geq 2$ then it is well known that d given by

$$d(x_n)_n,(y_n)_n) := \begin{cases} 0 & \forall\, n : x_n = y_n \\ (\min\{k \mid x_k \neq y_k\})^{-1} & \text{otherwise} \end{cases}$$

is a non-Archimedean metric on X, and it is easily seen that $(X,\mathbf{U}(d_\varepsilon))$ and $(X,\mathbf{U}(d_{\varepsilon'}))$ are not isomorphic if $\varepsilon \neq \varepsilon'$.

3. It is evident that the properties of $\mathbf{U}(d_\varepsilon)$ can be obtained from the corresponding ones of $\mathbf{U}(d)$ by replacing everywhere d by d_ε. So for instance, it follows from Proposition 3.4.1° that a basis for $\iota_{u,\alpha}(\mathbf{U}(d_\varepsilon))$ is given by $\{D'_r \mid \alpha < r\}$ where $D'_r = \{(x,y) \mid d(x,y) < \varepsilon r\}$.
Since this translation of properties of $\mathbf{U}(d)$ into properties of $\mathbf{U}(d_\varepsilon)$ is a simple exercise, while the formulation of the former is simpler, we shall continue to restrict ourselves mainly to the case $d \leq 1$.

4. From Proposition 3.3.6° it follows that all elements of $t(\mathbf{U}(d))$ are 1-Lipschitz, the converse however is not true. Consequently $t(\mathbf{U})$ is strictly coarser than the structure $\Delta(1)$ of [7].

PROPOSITION 3.5. The following hold :

1° $\varepsilon' \leq \varepsilon \Rightarrow \mathbf{U}(d_\varepsilon) \subset \mathbf{U}(d_{\varepsilon'})$;

2° $\inf_{\varepsilon\in\mathbb{R}_0^+} \mathbf{U}(d_\varepsilon) = \{1\}$;

3° $\sup_{\varepsilon\in\mathbb{R}_0^+} \mathbf{U}(d_\varepsilon) = \omega_u(\mathbf{U}_d)$.

PROOF. 1° is evident, and for 2° it suffices to remark that if $\mu \in \inf_{\varepsilon\in\mathbb{R}_0^+} \mathbf{U}(d_\varepsilon)$ then $\mu \geq 1 - \frac{1}{\varepsilon} d$ for all $\varepsilon \in \mathbb{R}_0^+$.
For 3° note that by Proposition 3.3.3° for all $\varepsilon \in \mathbb{R}_0^+$ we have $\mathbf{U}(d_\varepsilon) \subset \omega_u(\mathbf{U}_d)$ i.e. $\sup_{\varepsilon\in\mathbb{R}_0^+} \mathbf{U}(d_\varepsilon) \subset \omega_u(\mathbf{U}_d)$. The converse inclusion follows at once upon remarking that for all $\varepsilon \in \mathbb{R}_0^+$ we have $1_{D_\varepsilon} \in \mathbf{U}(d_\varepsilon)$ and that $\{D_\varepsilon \mid \varepsilon \in \mathbb{R}_0^+\}$ is a basis for $\mathbf{U}_d$. ∎

4. CONTINUITY AND CONVERGENCE

PROPOSITION 4.1. A map $f : (X,\mathbf{U}(d)) \to (X',\mathbf{U}(d'))$ is uniformly continuous if and only $f : (X,d) \to (X',d')$ is 1-Lipschitz, i.e. non-expansive.

PROOF. Immediate from the fact that $\psi \leq (f\times f)^{-1}(\psi')$ if and only if $d' \circ (f\times f) \leq d$. ∎

Since in the case of the above result the local character of the Lipschitz condition has disappeared we reformulate the foregoing result in the general case. With Remarks 3.4.2 and 3.4.3 in mind, the proof is obvious.

COROLLARY 4.2. A map $f : (X,\mathbf{U}(d_\varepsilon)) \to (X',\mathbf{U}(d'_{\varepsilon'}))$ is uniformly continuous if and only if $f : (X,d) \to (X',d')$ is ε-locally $\frac{\varepsilon'}{\varepsilon}$-Lipschitz. ∎

For concepts and results concerning convergence we refer to [2], [3].

PROPOSITION. 4.3. If $\mathbf{F}$ is a filter on X then $\mathbf{F} \to x$ in $(X,\mathbf{T}_d)$ if and only if $\lim \omega(\mathbf{F}) = \psi\langle x\rangle$ in $(X,t(\mathbf{U}(d)))$.

PROOF. As Theorem 5.3 in [7].

In spite of Remark 3.4.2 in special cases the spaces $(X,\mathbf{U}(d_\varepsilon))$ and $(X,\mathbf{U}(d_\varepsilon'))$ can be isomorphic.

PROPOSITION 4.4. If X is a non-Archimedean normed space then all $(X,\mathbf{U}(d_\varepsilon))$, $\varepsilon \in \mathbb{R}_0^+$, are mutually isomorphic.

PROOF. As Theorem 5.2 in [7]. ∎

5. COMPACTNESS.

For concepts and results on compactness and precompactness we refer to [6], [9].

THEOREM 5.1. The following are equivalent :

1° $(X,\mathbf{U}(d))$ is compact

2° $(X,\mathbf{U}(d))$ is precompact

3° (X,d) is totally bounded.

PROOF. The implications 1° => 2° => 3° are trivial. To show 3° => 1° let $\varepsilon \in I_0$. If $Y \subset X$ is a finite subset such that $X = \bigcup_{x\in Y} B(x,\varepsilon)$ then we have $\inf_{x\in Y} d(x,t) < \varepsilon$ for all $t \in X$ which is equivalent to $\sup_{x\in Y} \psi\langle x\rangle > 1 - \varepsilon$ which by Theorem 2.2 in [6] proves our claim. ∎

REMARK 5.2. Since for any $\varepsilon \in \mathbb{R}_0^+$ we have that d_ε is totally bounded if and only if d is totally bounded, it follows from the foregoing result that either all spaces $(X,\mathbf{U}(d_\varepsilon))$ are compact or none of them is.

6. COMPLETENESS.

For concepts and results concerning completeness and completions we refer to [9]. The following result is an immediate consequence of Theorem 4.5 in [9] and Proposition 3.3.3°.

THEOREM 6.1. The following are equivalent :

1° (X,d) is complete

2° $(X,\mathbf{U}(d))$ is ultracomplete. ∎

Given (X,d) we can now construct the following completions.

I. $(\tilde{X},\tilde{d})$ the metric completion of (X,d)

II. $(\hat{X},\mathbf{U}(\hat{d}))$ the ultracompletion of $(X,\mathbf{U}(d))$

III. $(X^*,\omega_u(\hat{\mathbf{U}}_d))$ the ultracompletion of $(X,\omega_u(\mathbf{U}_d))$.

Then we obtain the following collection of complete or ultracomplete spaces.

IV. The complete space $(\hat{X}, \iota_u(\hat{\mathbf{U}}(d)))$.

V. The complete space $(X^*, \iota_u(\hat{\omega}_u(\mathbf{U}_d)))$.

VI. The ultracomplete space $(\hat{X}, \mathbf{U}(\tilde{d}))$.

VII. The ultracomplete space $(\tilde{X}, \omega_u(\mathbf{U}_{\tilde{d}}))$.

Now it follows from [9] that

1° $(\tilde{X}, \omega_u(\mathbf{U}_{\tilde{d}}))$ and $(X^*, \omega_u(\hat{\mathbf{U}}_d))$ are isomorphic

2° $(\tilde{X}, \mathbf{U}_{\tilde{d}})$, $(\hat{X}, \iota_u(\hat{\mathbf{U}}(d))$ and $(X^*, \iota_u(\omega_u(\hat{\mathbf{U}}_d)))$ are isomorphic

Using the methods of [9] it can be shown conceptually that the remaining spaces $(\tilde{X}, \mathbf{U}(\tilde{d}))$ and $(\hat{X}, \mathbf{U}(\hat{d}))$ are isomorphic too.
However, we prefer to explicitly describe the isomorphism which at the same time allows us to describe the points of $\hat{X}$ too.

Given the non-Archimedean space (X,d), its metric completion $(\tilde{X}, \tilde{d})$ can be considered as the set $\tilde{X}$ of all equivalence classes of equivalent Cauchy sequences in X, equipped with the metric $\tilde{d}$ defined by $\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n)$, where $(x_n)_n$ and $(y_n)_n$ are arbitrary representatives of $\tilde{x}$ and $\tilde{y}$ respectively
The ultracompletion [9] of $(X, \mathbf{U}(d))$ is given by $(\hat{X}, \mathbf{U}(\hat{d}))$, where $\hat{X}$ is the set of all minimal hyper Cauchy prefilters on $(X, \mathbf{U}(d))$, and where $\mathbf{U}(\hat{d})$ is the fuzzy uniformity generated by $\{\hat{\psi}\}$, this function being defined by

$$\hat{\psi}(\mathbf{C}_1, \mathbf{C}_2) = 1 - \inf\{\varepsilon \mid \exists\ \mu_\varepsilon \in \mathbf{C}_1 \cap \mathbf{C}_2 : \mu_\varepsilon \times \mu_\varepsilon \leq \psi + \varepsilon\}.$$

<u>LEMMA</u> 6.2. If $(x_n)_n$ is a Cauchy-sequence in (X,d), the sequence $(\psi\langle x_n\rangle)_n$ converges uniformly to a mapping $\gamma(\tilde{x}) : X \to I : t \to \lim_{n\to\infty} \psi(t, x_n)$ which depends only on the equivalence class $\tilde{x}$ of $(x_n)_n$, and which has the following properties :

a. $\sup_{t\in X} \gamma(\tilde{x})(t) = 1$,

b. $\gamma(\tilde{x}) \times \gamma(\tilde{x}) \leq \psi$,

c. $\psi\langle\gamma(\tilde{x})\rangle = \gamma(\tilde{x})$.

<u>PROOF</u>. If $\tilde{t}$ is the class of the constant sequence $(t_n = t)_n$, we know that $\tilde{d}(\tilde{t}, \tilde{x}) = \lim_{n\to\infty} d(t, x_n)$ is independent of the choice of $(x_n)_n \in \tilde{x}$.
If $\varepsilon > 0$ and n_0 is chosen such that

$$p \geq n_0,\ q \geq n_0 \Rightarrow \psi(x_p, x_q) \geq 1 - \varepsilon,$$

then for $x \in X$, $p \geq n_0$, $q \geq n_0$ either $\psi(x, x_p) < 1 - \varepsilon$ and then $\psi(x, x_q) = \psi(x, x_p)$, or $\psi(x, x_p) \geq 1 - \varepsilon$ and then also $\psi(x, x_q) \geq 1 - \varepsilon$, so in any case $|\psi(x, x_p) - \psi(x, x_q)| \leq \varepsilon$, which proves the uniform convergence. The property a follows by considering $\gamma(\tilde{x})(x_n)$, and b and c follow by standard verification. ∎

It follows from the foregoing lemma, that the prefilter $\Gamma(\tilde{x}) = \{\gamma(\tilde{x})\}^{\sim} = \dot{\gamma}(\tilde{x})$ is a minimal hyper Cauchy prefilter on $(X, \mathbf{U}(d))$, and so we obtain a mapping $\Gamma : \tilde{X} \to \hat{X}$.

<u>LEMMA</u> 6.3. If $\mathbf{C}$ is a hyper Cauchy prefilter on $(X, \mathbf{U}(d))$ then there exists a Cauchy sequence $(x_n)_n$ in (X,d) such that

a. $\forall\, n \in \mathbb{N} : \gamma_n := \sup_{k \geq n} \psi\langle x_k \rangle \in \mathbf{C}$;

b. $(\gamma_n)_n$ converges uniformly to $\gamma(\tilde{x}) \in \mathbf{C}$, where $\tilde{x}$ is the equivalence class of $(x_n)_n$;

c. $\forall\, n \in \mathbb{N} : \gamma_n \times \gamma_n \leq \psi + \rho_n$ where $\lim_{n\to\infty} \rho_n = 0$.

PROOF. It follows from (HC3) that we can find a non-increasing sequence $(\beta_n)_n$ of elements of $\mathbf{C}$ such that for all $n \in \mathbb{N}$:

$$\beta_n \times \beta_n \leq \psi + 2^{-n}. \tag{1}$$

By (HC1) we can find a sequence $(x_n)_n$ in X such that for all $n \in \mathbb{N}$:

$$1 - 2^{-n-1} \leq \beta_n(x_n). \tag{2}$$

Since $(\beta_n)_n$ is non-increasing it follows that

$$1 - 2^{-n-2} \leq \beta_n(x_{n+1})$$

and consequently

$$1 - 2^{-n-1} \leq \beta_n(x_n) \wedge \beta_n(x_{n+1}) \leq \psi(x_n, x_{n+1})$$

which shows $(x_n)_n$ is a Cauchy sequence.
Further by (1) and (2) we have that for all $n \in \mathbb{N}$ and $x \in X$:

$$\begin{aligned} \beta_k(x) - 2^{-k} &\leq \beta_k(x) \wedge (1-2^{-k-1}) - 2^{-k-1} \\ &\leq \beta_k(x) \wedge \beta_k(x_k) - 2^{-k-1} \\ &\leq \psi(x_k, k). \end{aligned}$$

Thus it follows from (HC2) that for all $n \in \mathbb{N}$:

$$\gamma_n := \sup_{k \geq n} \psi\langle x_k \rangle \in \mathbf{C}.$$

Since $(\psi\langle x_n \rangle)_n$ converges uniformly to $\gamma(\tilde{x})$ the same is true for $(\gamma_n)_n$ and thus again by (HC2) we obtain that $\gamma(\tilde{x}) \in \mathbf{C}$.

Finally we still have that for all $n \in \mathbb{N}$ and $x, y \in X$:

$$\begin{aligned} \gamma_n(x) \wedge \gamma_n(y) &= \sup_{k \geq n, m \geq n} \psi(x_k, x) \vee \psi(x_m, y) \\ &\leq \sup_{k \geq n, m \geq n} \psi(x_k, x) \wedge \psi(x_m, y) \wedge (\psi(x_k, x_m) + 2^{-n-1}) \\ &\leq \psi(x, y) + 2^{-n-1}. \end{aligned}$$

■

We are now in a position to prove the isomorphism result.

THEOREM 6.4. The map

$$\Gamma : (\tilde{X}, \mathbf{U}(\tilde{d})) \to (\hat{X}, \mathbf{U}(\hat{d}))$$

is an isomorphism.

PROOF. To see that Γ is into, let $\tilde{x}, \tilde{y} \in \tilde{X}$, $\tilde{x} \neq \tilde{y}$ and let $(x_n)_n$ and $(y_n)_n$ be representatives of $\tilde{x}$ and $\tilde{y}$ respectively. Then there exists $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for all $p, q \geq n_0$: $d(x_p, y_q) \geq \varepsilon$ which implies that for any $p \geq n_0$ we have $\gamma(\tilde{y})(x_p) \leq 1 - \varepsilon$

whereas $\lim_{n\to\infty} \gamma(\tilde{x})(x_n) = 1$. Thus $\gamma(\tilde{y}) \neq \gamma(\tilde{x})$ and therefore $\Gamma(\tilde{y}) \neq \Gamma(\tilde{x})$.
To see that Γ is onto, take $\mathbf{C} \in \hat{X}$, and consider $\gamma(\tilde{x})$ as constructed in Lemmas 6.2 and 6.3. It then follows that $\Gamma(\tilde{x}) \subset \mathbf{C}$, and so $\mathbf{C} = \Gamma(\tilde{x})$ by minimality.
To show that Γ is an isomorphism it now suffices to show that $\hat{\psi}_d \circ (\Gamma\times\Gamma) = \psi_{\tilde{d}}$. Since $\Gamma(\tilde{x}) = \dot{\gamma}(\tilde{x})$ we first have

$$\hat{\psi}_d(\Gamma(\tilde{x}),\Gamma(\tilde{y})) = 1 - \sup_{s,t\in X} ((\gamma(\tilde{x})\vee\gamma(\tilde{y})) \times (\gamma(\tilde{x})\vee\gamma(\tilde{y}))(s,t)-\psi(s,t)).$$

Using distributivity in

$$(\gamma(\tilde{x})\vee\gamma(\tilde{y}))(s) \wedge (\gamma(\tilde{x})\vee\gamma(\tilde{y}))(t)$$

the symmetry of ψ and the fact that $\gamma(\tilde{x}) \times \gamma(\tilde{y}) \leq \psi$, we obtain

$$\hat{\psi}_d(\Gamma(\tilde{x}),\Gamma(\tilde{y})) = 1 - \sup_{s,t\in X} (\gamma(\tilde{x})(s)\vee\gamma(\tilde{y})(t)-\psi(s,t))$$

and from this it follows that in the end we have to show that for any pair of Cauchy sequences $(x_n)_n$ and $(y_n)_n$ in (X,d) we have :

$$\lim d(x_n,y_n) = \sup_{s,t\in X} \lim_{n\to\infty}(d(s,t)-d(x_n,s)\vee d(y_n,t)).$$

From the ultrametric property we obtain

$$\begin{aligned}
&\lim_{n\to\infty}(d(s,t)-d(x_n,s)\vee d(y_n,t)) \\
&\quad \leq \lim_{n\to\infty}(d(x_n,y_n)\vee d(x_n,s)\vee d(y_n,t)-d(x_n,s)\vee d(y_n,t)) \\
&\quad = \lim_{n\to\infty}(d(x_n,y_n)-d(x_n,s)\vee d(y_n,t)) \vee 0 \\
&\quad \leq \lim_{n\to\infty} d(x_n,y_n)
\end{aligned}$$

while on the other hand, since $\sup : I^{X\times X} \to I$ is continuous if $I^{X\times X}$ is equipped with the uniform topology and I with the usual one, we have

$$\begin{aligned}
&\sup_{s,t\in X} \lim_{n\to\infty}(d(s,t)-d(x_n,s)\vee d(y_n,t)) \\
&\quad = \lim_{n\to\infty} \sup_{s,t\in X} (d(s,t)-d(x_n,s)\vee d(y_n,t)) \\
&\quad \geq \lim_{n\to\infty} d(x_n,y_n).
\end{aligned}$$

■

In order to describe the points of $\hat{X}$ in more detail we have the next result.

<u>THEOREM</u> 6.5. The following are equivalent :

1° $\mathbf{C}$ is a minimal hyper Cauchy prefilter on $(X,\mathbf{U}(d))$

2° $\mathbf{C} = \dot{\theta}$ where θ fulfils :

a. $\theta \times \theta \leq \psi$,

b. $\psi\langle\theta\rangle = \theta$,

c. $\sup \theta = 1$;

3° $\mathbf{C}$ is a prefilter with a basis $\{\gamma_n | n \in \mathbf{N}\}$ fulfilling :

a. $\forall\, n \in \mathbb{N} : \gamma_{n+1} \leq \gamma_n$

b. $\forall\, n \in \mathbb{N} : \psi\langle\gamma_n\rangle = \gamma_n$

c. $\forall\, n \in \mathbb{N}, \exists\, x_n \in X : \gamma_n(x_n) = 1$

d. $\forall\, n \in \mathbb{N} : \gamma_n \times \gamma_n \leq \psi + \rho_n$ where $\lim_{n\to\infty} \rho_n = 0$.

PROOF. Since 1° => 2° was proved in Theorem 6.4 and Lemma 6.2, while 2° => 1° and 3° => 2° are obvious, it is sufficient to prove 2° => 3°.
We can repeat the construction in Lemma 6.3 with $\beta_n = \theta$ for all $n \in \mathbb{N}$. The sequence $(\gamma_n)_n$ has the properties a, c and d by the construction in Lemma 6.3. As to b, this follows from

$$\begin{aligned}\psi\langle\gamma_n\rangle(x) &= \sup_{t\in X} \gamma_n(t) \vee \psi(t,x) = \sup_{t\in X} \sup_{k\geq n}(\mu_k(t)\wedge\psi(t,x))\\ &= \sup_{k\geq n} \sup_{t\in X}(\mu_k(t)\wedge\psi(t,x)) = \sup_{k\geq n} \psi\langle\mu_k\rangle(x)\\ &= \sup_{k\geq n} \mu_k(x) = \gamma_n(x),\end{aligned}$$

where $\mu_n = \psi\langle x_n\rangle$.
Since the minimal hyper Cauchy prefilter generated by $\{\gamma_n | n \in \mathbb{N}\}$ is coarser than $\mathbf{C}$ it coincides with $\mathbf{C}$. ∎

REMARK 6.6. A characterization of minimal Cauchy filters, probably belonging to the folklore of the subject, and with a standard proof which we leave to the reader, is given by the following ((X,d) is a pseudometric space) : $\mathbf{F}$ is a minimal Cauchy filter on $(X,\mathbf{U}_d)$ if and only if $\mathbf{F}$ is a filter having a basis $(B_n)_{n\in\mathbb{N}}$ which is a non-increasing chain of open balls $B_n = B(x_n,r_n)$ with the property $\lim_{n\to\infty} r_n = 0$. An alternative method for proving the isomorphism of $(\hat{X},\mathbf{U}(\hat{d}))$ and $(\tilde{X},\mathbf{U}(\tilde{d}))$ can be based on this and on Theorem 6.5. Indeed, we consider $\tilde{X}$ as the set of minimal Cauchy filters on (X,d), and the foregoing then allows a bijection between minimal hyper Cauchy prefilters on $(X,\mathbf{U}(d))$ and minimal Cauchy filters on (X,d).

7. CONNECTEDNESS

In [8] a number of connectedness concepts in G. Preuss' sense have been introduced and studied.

We recall that a space $(X,\Delta) \in |FTS|$ is called 2_α-connected if and only if there does not exist a non-empty proper subset $A \subset X$ such that $\{\alpha 1_A, \alpha 1_{X\setminus A}\} \subset \Delta$ and is called D-connected if and only if it is 2_α-connected for each $\alpha \in I_0$. For the meaning of the notations 2_α and D we refer to [8].

PROPOSITION 7.1. For $\alpha \in I_0$ and $A \in 2^X\setminus\{\phi,X\}$ the following are equivalent :

1° $\{\alpha 1_A, \alpha 1_{X\setminus A}\} \subset t(\mathbf{U}(d))$

2° $d(A,X\setminus A) \geq \alpha$.

PROOF. This follows by straightforward verification using e.g. Proposition 3.4.6. [8]. ∎

The following is an immediate consequence.

THEOREM 7.2. The following hold :

1° $(X,t(\mathbf{U}(d)))$ is 2_α-connected if and only if there exists no non-empty proper subset $A \subset X$ such that $d(A,X\backslash A) \geq \alpha$;

2° $(X,t(\mathbf{U}(d)))$ is D-connected if and only if there exists no non-empty proper subset of $A \subset X$ such that $d(A,X\backslash A) > 0$. ∎

8. CATEGORICAL CONSIDERATIONS

Let NA(1) stand for the category of non-Archimedean pseudometric spaces (X,d) where $d \leq 1$ and with morphisms 1-Lipschitz or non-expansive maps. We already know that the functor

$$\begin{aligned} NA(1) &\longrightarrow FUS \\ (X,d) &\longrightarrow (X,\mathbf{U}(d)) \end{aligned}$$

which leaves morphisms unaltered is a full embedding. Consequently we may consider NA(1) to be a full subcategory of FUS. We shall now prove that NA(1) actually is a very nice subcategory (see also [1]).

THEOREM 8.1. NA(1) is a bicoreflective subcategory of FUS.

PROOF. Given $(X,\mathbf{U}) \in |FUS|$ put

$$\psi_{\mathbf{U}}(x,y) := \inf_{\nu\in\mathbf{U}} \nu(x,y).$$

Obviously $\psi_{\mathbf{U}} \circ \psi_{\mathbf{U}} = \psi_{\mathbf{U}}$, $\psi_{\mathbf{U}}$ is symmetric and $\forall\, x \in X : \psi_{\mathbf{U}}(x,x) = 1$. Thus $\{\psi_{\mathbf{U}}\}$ generates a fuzzy uniformity $\mathbf{U}^*$. SInce $d_{\mathbf{U}} := 1 - \psi_{\mathbf{U}}$ clearly is a non-Archimedean pseudometric, we moreover have $\mathbf{U}^* = \mathbf{U}(d_{\mathbf{U}})$. Since $\mathbf{U}^* \supset \mathbf{U}$ it is also immediately clear that

$$id_X : (X,\mathbf{U}^*) \to (X,\mathbf{U})$$

is uniformly continuous. Now, given $(Y,\mathbf{W}) \in |NA(1)|$ and a uniformly continuous map

$$f : (Y,\mathbf{W}) \longrightarrow (X,\mathbf{U})$$

we can choose a non-Archimedean pseudometric $d \leq 1$ such that $\mathbf{W} = \{\psi_d\}^\sim$ and it then follows that for all $\nu \in \mathbf{U} : \psi_d \leq \nu \circ (f\times f)$ and thus also $\psi_d \leq \psi_{\mathbf{U}} \circ (f\times f)$ which proves that

$$f : (Y,\mathbf{W}) \longrightarrow (X,\mathbf{U})^*$$

if also uniformly continuous. ∎

REMARKS 8.2. 1) In [15] it was shown that for $(X,\mathbf{U}) \in |FUS|$ the T_0-, T_1-, and T_2-separation functions τ_0, τ_1, τ_1' and τ_2 on $(X,t(\mathbf{U}))$ are given by

$$\tau_0(x,y) = \tau_1(x,y) = \tau_1'(x,y) = \tau_2(x,y) = 1 - \inf_{\nu\in\mathbf{U}} \nu(x,y).$$

Thus we simply have

$$\tau_0 = \tau_1 = \tau_1' = \tau_2 = d_{\mathbf{U}}.$$

2) It is easily seen that NA(1) is not a reflective subcategory of FUS. If $(X_j,\mathbf{U}(d_j))_{j\in J}$ is a non-finite collection in $|NA(1)|$ then their product is given by $(\prod_{j\in J} X_j,\mathbf{U})$ where $\mathbf{U}$ is the fuzzy uniformity generated by

$$\{\psi_K | K \subset J,\ K \text{ finite}\}$$

and where ψ_K is defined by

$$\psi_K : (\prod_{j\in J} X_j) \times (\prod_{j\in J} X_J) \longrightarrow I$$

$$((x_j)_j,(y_j)_j) \longrightarrow \inf_{k\in K} \psi_{d_k}(x_k,y_k).$$

Clearly, then $(\prod_{j\in J} X_j, \mathbf{U}) \notin |NA(1)|$.
NA(1) is however closed for finite products in FUS.

9. DETERMINATION OF $\mathbf{U}(d)$ BY ITS LEVEL UNIFORMITIES

We recall [10], [11] that a uniformity $\mathbf{U}$ on X is called non-Archimedean if there exists a collection Φ of partitions of X such that $\{\bigcup_{P\in\mathbf{P}} P \times P | \mathbf{P} \in \Phi\}$ is a basis for $\mathbf{U}$.

In the sequel, if $\mathbf{P}$ is a partition of X, we shall write $\mathbf{P}(x)$ for the member of $\mathbf{P}$ that contains $x \in X$.

PROPOSITION 9.1. $\iota_{u,\alpha}(\mathbf{U}_d)$ is a non-Archimedean uniformity on X, generated by $\Phi := \{\mathbf{P}_r | r > \alpha\}$, where $\mathbf{P}_r := \{B(x,r) | x \in X\}$.

PROOF. Since $\mathbf{P}_r(x) = B(x,r)$, we have $\mathbf{P}_r(x) \times \mathbf{P}_r(x) = \{(y,z) | \exists\ x \in X : d(x,y) \vee d(x,z) < r\} = \{(y,x) | d(y,z) < r\} = D_r$, and it follows from Proposition 3.3.1° that $\iota_{u,\alpha}(\mathbf{U}_d)$ is non-Archimedean.
The rest of the theorem is a reformulation in this particular case of well-known relations [12] between diagonal and covering uniformities and the fact that $\mathbf{P}_r < \mathbf{P}_s$ if $r < s$ and $\mathbf{P}^* = \mathbf{P}$. ∎

An immediate consequence of this is the next result.

PROPOSITION 9.2. There exists a family $(\mathbf{P}_\alpha)_{\alpha\in I_I}$ of partitions of X, satisfying the condition

$$\alpha < \beta \Rightarrow \mathbf{P}_\alpha < \mathbf{P}_\beta$$

and such that $\iota_{u,\alpha}(\mathbf{U}(d))$ is generated by the family $(\mathbf{P}_\beta)_{\beta>\alpha}$ of coverings, i.e. such that

$$\{\bigcup_{P\in\mathbf{P}_\beta} P \times P | \beta > \alpha\}$$

is a basis for $\iota_{u,\alpha}(\mathbf{U}(d))$.

But we also have the converse.

THEOREM 9.3. If $(\mathbf{P}_\alpha)_{\alpha\in I_1}$ is a family of partitions of X, satisfying the condition

$$\alpha < \beta \Rightarrow \mathbf{P}_\alpha < \mathbf{P}_\beta, \tag{1}$$

then there is a non-Archimedean pseudometric $d \leq 1$ on X, such that for each $\alpha \in I_1$ the uniformity $\mathbf{U}_\alpha$, generated by $(\mathbf{P}_\beta)_{\beta>\alpha}$ is the α-level-uniformity $\iota_{u,\alpha}(\mathbf{U}(d))$ of $\mathbf{U}(d)$.

PROOF. We first remark that by (1)

$$\mathbf{P}_\alpha(x) = \mathbf{P}_\alpha(y) \text{ and } \alpha < \beta \Rightarrow \mathbf{P}_\beta(x) = \mathbf{P}_\beta(y).$$

We can therefore define $d : X \times X \to I$ by

$$d(x,y) := \inf\{\alpha \mid \mathbf{P}_\alpha(x)=\mathbf{P}_\alpha(y)\} = \sup\{\alpha \mid \mathbf{P}_\alpha(x)\neq\mathbf{P}_\alpha(y)\}$$

(with $\inf \emptyset = 1$, $\sup \emptyset = 0$). Clearly $d(x,x) = 0$ and $d(x,y) = d(y,x)$. Further, if

$$d(x,y) = \alpha' \leq \alpha'' = d(x,z),$$

then for all $\alpha > \alpha''$ we have $\mathbf{P}_\alpha(z) = \mathbf{P}_\alpha(x) = \mathbf{P}_\alpha(y)$, and therefore $d(y,z) \leq \alpha''$. So d is a non-Archimedean pseudometric, and we only have to prove that $\mathbf{U}_d = \iota_{u,\alpha}(\mathbf{U}(d))$. First, if $\alpha < \beta$, we can take r such that $\alpha < r < \beta$. If then $d(x,y) < r$, we have $\mathbf{P}_\beta(x) = \mathbf{P}_\beta(y)$, so $(x,y) \in \bigcup_{P\in\mathbf{P}_\beta} P \times P$, and therefore $D_r \subset \bigcup_{P\in\mathbf{P}_\beta} P \times P$. From Proposition 9.2 it now follows that $\bigcup_{P\in\mathbf{P}_\beta} P \times P \in \iota_{\alpha,u}(\mathbf{U}(d))$, whence $\mathbf{U}_\alpha \subset \iota_{u,\alpha}(\mathbf{U}(d))$ by arbitrariness of $\beta > \alpha$. Conversely, if $\alpha < r$, we can take $\alpha < \beta < r$ and then

$$d(x,y) \geq r \Rightarrow \mathbf{P}_\beta(x) \neq \mathbf{P}_\beta(y) \Rightarrow (x,y) \notin \bigcup_{P\in\mathbf{P}_\beta} P \times P,$$

so $\bigcup_{P\in\mathbf{P}_\beta} P \times P \subset D_r$, whence $D_r \in \mathbf{U}_\alpha$. Again, by arbitrariness of $r > \alpha$, we obtain $\iota_{\mu,\alpha}(\mathbf{U}(d)) \subset \mathbf{U}_\alpha$, and so we are done. ∎

REFERENCES

1. HERRLICH, H. Categorical Topology 1971-1981, Gen. Topol. Rel. Modern Anal. and Algebra V (1982), 279-383.
2. LOWEN, E., LOWEN, R. A characterization of convergence in fuzzy topological spaces, Internat. J. Math. & Math. Sci. 8 (1985), 497-511.
3. LOWEN, R. Convergence in fuzzy topological spaces, General Topol. Appl. 10 (1979), 149-160.
4. LOWEN, R. Fuzzy Uniform Spaces, J. Math. Anal. Appl. 82 (1981), 370-385.
5. LOWEN, R. Fuzzy Neighborhood Spaces, Fuzzy Sets and Systems 7 (1982), 165-189.
6. LOWEN, R. Compactness Notions in Fuzzy Neighborhood Spaces, Manuscripta Math. 38 (1982), 265-287.
7. LOWEN, R. Metric Spaces Viewed as Fuzzy Topological Spaces Induced by Lipschitz Functions, Math. Nachr. 120 (1985), 249-265.
8. LOWEN, R, SRIVASTAVA A.K. On Preuss' connectedness concept in FTS, submitted.
9. LOWEN, R., WUYTS, P. Completeness, Compactness and Precompactness in Fuzzy Uniform Spaces, Part I, J. Math. Anal. Appl. 90 (1982), 563-581; Part II, J. Math. Anal. Appl. 92 (1983), 342-371.
10. SCHIKHOF, W. Ultrametric calculus, Cambridge U.P., Cambridge, 1984.
11. VAN ROOIJ, A. Non-Archimedean Functional Analysis, Marcel Dekker, New York, 1978.
12. WILLARD, S. General Topology, Addison-Wesley, Reading, 1970.
13. WUYTS, P. On the determination of fuzzy topological spaces and fuzzy neighborhood spaces by their level topologies, Fuzzy Sets and Systems 12 (1984), 71-86.
14. WUYTS, P., LOWEN, R. On separation Axioms in Fuzzy Topological Spaces, Fuzzy Neighborhood Spaces, and Fuzzy Uniform Spaces, J. Math. Anal. Appl. 93 (1983), 27-41.
15. WUYTS, P., LOWEN, R. On local and global measures of separation in fuzzy topological spaces, Fuzzy Sets and Systems 19 (1986), 51-80.