Internat. J. Math. & Math. Sci.
VOL. 12 NO. 3 (1989) 493-502

A GENERALIZATION OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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(Received January 6, 1987)

ABSTRACT. Let T_p be the class of analytic and p-valent functions which can be expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}, \quad |z| < 1.$$

The subclasses $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$ of T_p have been considered. Sharp results concerning coefficient estimates, distortion and covering theorems are obtained. The radius of convexity for the class $T_p^*(A,B,\alpha)$ is determined. It is further proved that the classes $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$ are closed under arithmetic mean and convex linear combinations.

KEY WORDS AND PHRASES. P-valent, Analytic, Radius of Convexity.
1980 AMS SUBJECT CLASSIFICATION CODES. 30A32, 30A36.

1. INTRODUCTION.

Let $S_p (p \geq 1)$ denote the class of functions of the form

$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ which are analytic and p-valent in the unit disc

$U = \{z: |z| < 1\}$. A function f is said to be subordinate to a function $F (f \prec F)$ if there exists an analytic function $\phi(z)$ with $|\phi(z)| \leq |z|$, $z \in U$, such that $f = f.\phi$.

For A,B fixed, $-1 \leq A < B \leq 1$, and $0 \leq \alpha < p$, we say that $f \in S_p^*(A,B,\alpha)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{p + [pB+(A-B)(p-\alpha)]z}{1 + Bz}, \quad z \in U,$$

or equivalently $f \in S_p^*(A,B,\alpha)$ if and only if

$$\left|\frac{\frac{zf'(z)}{f(z)} - p}{B\frac{zf'(z)}{f(z)} - [pB+(A-B)(p-\alpha)]}\right| < 1, \quad z \in U.$$

Further f is said to belong to the class $K_p(A,B,\alpha)$ if and only if $\frac{zf'(z)}{p} \in S_p^*(A,B,\alpha)$.

Let T_p denote the subclass of S_p consisting of functions analytic and p-valent which can be expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n}, \text{ and we set}$$

$$T_p^*(A,B,\alpha) = S_p^*(A,B,\alpha) \quad T_p \text{ and } C_p(A,B,\alpha) = Kp(A,B,\alpha) \quad T_p.$$

Silverman [3], Gupta and Jain [2] and Silverman and Silvia [4,5] have studied Certain subclasses of univalent functions with negative coefficients. Also Goel and Sohi [1] have studied certain subclasses of multivalent functions with negative coefficients. In this paper we obtain coefficient estimates, distortion and covering theorems for the classes $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$. We also determine the radius of convexity for the class $T_p^*(A,B,\alpha)$. It is further shown that the classes $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$ are closed under arithmetic mean and convex linear combinations. By taking $\alpha=0$, we get results due to Goel and Sohi [1] and by assigning specific values to A and B and taking p=1, we get results due to Silverman [3] and Gupta and Jain [2].

2. COEFFICIENT INEQUALITIES.

THEOREM 1. A function $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n}$ is in $T_p^*(A,B,\alpha)$ if and only if

$$\sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-\alpha)]|a_{p+n}| \leq (B-A)(p-\alpha). \tag{2.1}$$

The result is sharp.

PROOF. Let $|z| = 1$, then

$$\begin{aligned}
&|zf'(z) - pf(z)| - |B\, zf'(z) - [pB+(A-B)(p-\alpha)]\, f(z)| \\
&= \left|\sum_{n=1}^{\infty} -n\, |a_{p+n}|\, z^{p+n}\right| - |(B-A)(p-\alpha)z^p \\
&\quad - \sum_{n=1}^{\infty} [nB + (B-A)(p-\alpha)]\, |a_{p+n}|\, z^{p+n}| \\
&\leq \sum_{n=1}^{\infty} [(1+B)n + (B-A)(p-\alpha)]|a_{p+n}| - (B-A)(p-\alpha) \leq 0.
\end{aligned}$$

Hence by the principle of maximum modulus $f(z) \in T_p^*(A,B,\alpha)$.

Conversely, suppose that

$$\left|\frac{\frac{zf'(z)}{f(z)} - p}{B.\frac{zf'(z)}{f(z)} - [pB + (A-B)(p-\alpha)]}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty} n|a_{p+n}|\, z^{p+n}}{(B-A)(p-\alpha)z^p - \sum_{n=1}^{\infty} [nB+(B-A)(p-\alpha)]|a_{p+n}|z^{p+n}}\right| < 1, \qquad z \in U.$$

Since $|\mathrm{Re}\, z| \leq |z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty} n\,|a_{p+n}|\, z^{p+n}}{(B-A)(p-\alpha)z^p - \sum_{n=1}^{\infty} [nB+(B-A)(p-\alpha)]|a_{p+n}|\, z^{p+n}}\right\} < 1. \qquad (2.2)$$

Choose values of z on the real axis so that $\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1$ through real values, we obtain

$$\sum_{n=1}^{\infty} n|a_{p+n}| \leq \{(B-A)(p-\alpha) - \sum_{n=1}^{\infty} [nB + (B-A)(p-\alpha)]\, |a_{p+n}|\}$$

which implies that

$$\sum_{n=1}^{\infty} [(1+B)n + (B-A)(p-\alpha)]\, |a_{p+n}| \leq (B-A)(p-\alpha).$$

The function

$$f(z) = z^p - \sum_{n=1}^{\infty} \frac{(B-A)(p-\alpha)}{(1+B)n + (B-A)(p-\alpha)} z^{p+n}$$

is an extremal function.

COROLLARY 1. If $f \in T_p^*(A,B,\alpha)$ then $|a_{p+n}| \leq \frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)}$, with

equality only for functions of the form $f(z) = z^p - \frac{(B-A)(p-\alpha)}{(1+B)n + (B-A)(p-\alpha)} z^{p+n}$.

COROLLARY 2. A function $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n}$ is in $C_p(A,B,\alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right) [(1+B)n+(B-A)(p-\alpha)]|a_{p+n}| \leq (B-A)(p-\alpha).$$

PROOF. It is well known that $f \in C_p(A,B,\alpha)$ if and only if $\frac{zf'(z)}{p} \in T_p^*(A,B,\alpha)$.

Since $\frac{zf'(z)}{p} = z^p - \sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right) |a_{p+n}|z^{p+n}$

we may replace $|a_{p+n}|$ with $\left(\frac{n+p}{p}\right) |a_{p+n}|$ in <u>Theorem 1</u>.

3. REPRESENTATION FORMULA.

THEOREM 2. A function $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$ is in $T_p^*(A,B,\alpha)$ if and only if

$$f(z) = z^p \exp\{(B-A)(p-\alpha) \int_0^z \frac{\phi(t)}{1-Bt\phi(t)} dt\}, \tag{3.1}$$

where $\phi(z)$ is analytic in U and satisfies $|\phi(z)| < 1$, $z \in U$.

PROOF. Let $f(z) \in T_p^*(A,B,\alpha)$, then

$$\left|\frac{\frac{zf'(z)}{f(z)} - p}{B\frac{zf'(z)}{f(z)} - [pB+(A-B)(p-\alpha)]}\right| < 1, \quad z \in U.$$

Since the absolute value vanishes for $z = o$, we have

$$\left|\frac{\frac{zf'(z)}{f(z)} - p}{B.\frac{zf'(z)}{f(z)} - [pB+(A-B)(p-\alpha)]}\right| = h(z) \tag{3.2}$$

where $h(z)$ is analytic in U and $|h(z)| < 1$ for $z \in U$. Integrating (3.2) with $h(z) = z\phi(z)$ we find that

$$f(z) = z^p. \exp\{(B-A)(p-\alpha) \int_o^z \frac{\phi(t)}{1-Bt\phi(t)} dt\}.$$

The converse is obtained by differentiating (3.1).

4. DISTORTION AND COVERING THEOREMS FOR $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$.

THEOREM 3. If $f(z) \in T_p^*(A,B,\alpha)$, then

$$r^p - \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^{p+1} \leq |f(z)| \leq$$

$$r^p + \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^{p+1} \quad (|z| = r), \tag{4.1}$$

with equality for $f(z) = z^p - \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} z^{p+1} (z=\pm r)$.

PROOF. From Theorem 1, we have

$$[1+B+(B-A)(p-\alpha)] \sum_{n=1}^{\infty} |a_{p+n}| \leq \sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-\alpha)]|a_{p+n}| \leq (B-A)(p-\alpha).$$

This implies that

$$\sum_{n=1}^{\infty} |a_{p+n}| \leq \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)}. \tag{4.2}$$

Thus

$$|f(z)| \leq |z|^p + \sum_{n=1}^{\infty} |a_{p+n}| \, |z|^{p+n}$$

$$\leq r^p(1 + r \sum_{n=1}^{\infty} |a_{p+n}|)$$

$$< r^p + \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^{p+1} .$$

Similarly,

$$|f(z)| > |z|^p - \sum_{n=1}^{\infty} |a_{p+n}| \cdot |z|^{p+n}$$

$$> r^p(1-r \sum_{n=1}^{\infty} |a_{p+n}|)$$

$$> r^p - \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^{p+1} .$$

COROLLARY 3. If $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} \in C_p(A,B,\alpha)$, then

$$r^p - \frac{(B-A)(p-\alpha)(p+1)}{p[1+B+(B-A)(p-\alpha)]} r^{p+1} < |f(z)| <$$

$$r^p + \frac{(B-A)(p-\alpha)(p+1)}{p[1+B+(B-A)(p-\alpha)]} r^{p+1} \quad (|z| = r),$$

with equality for

$$f(z) = z^p - \frac{(B-A)(p-\alpha)(p+1)}{p[1+B+(B-A)(p-\alpha)]} z^{p+1} (z = \pm r).$$

THEOREM 4. The disc $|z| < 1$ is mapped onto a domain that contains the disc $|w| < \frac{1 + B}{1+B+(B-A)(p-\alpha)}$ by any $f \in T_p^*(A,B,\alpha)$, and onto a domain that contains the disc $|w| < \frac{p+[pB+(A-B)(p-\alpha)]}{p.[1+B+(B-A)(p-\alpha)]}$ by any $f \in C_p(A,B,\alpha)$.

The theorem is sharp, with extremal functions

$$z^p - \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} z^{p+1} \in T_p^*(A,B,\alpha) \text{ and}$$

$$z^p - \frac{(B-A)(p-\alpha)(p+1)}{p[1+B+(B-A)(p-\alpha)]} z^{p+1} \in C_p(A,B,\alpha).$$

PROOF. The results follow upon letting $r \to 1$ in Theorem 3 and Corollary 3.

THEOREM 5. If $f \in T_p^*(A,B,\alpha)$, then

$$pr^{p-1} - \frac{(p+1)(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^p < |f'(z)| <$$

$$pr^{p-1} + \frac{(p+1)(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^p \quad (|z| = r).$$

Equality holds for

$$f(z) = z^p - \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} z^{p+1} \quad (z = \pm r).$$

PROOF. We have

$$|f'(z) \leq pr^{p-1} + \sum_{n=1}^{\infty} (p+n) \, |a_{p+n}| \, r^{p+n-1}$$

$$\leq pr^{p-1} + r^p \sum_{n=1}^{\infty} (p+n) \, |a_{p+n}|$$

$$= r^{p-1} \, [p+r \sum_{n=1}^{\infty} (p+n) \, |a_{p+n}|]. \tag{4.3}$$

In view of Theorem 1,

$$\sum_{n=1}^{\infty} (1+B) \, [n+p- \frac{p(1+B)+(A-B)(p-\alpha)}{1+B}] \, |a_{n+p}|$$

$$\leq (B-A)(p-\alpha)$$

or

$$\sum_{n=1}^{\infty} (1+B)(n+p) \, |a_{n+p}| \leq (B-A)(p-\alpha) +$$

$$[p(1+B)+(A-B)(p-\alpha)] \sum_{n=1}^{\infty} |a_{n+p}| \tag{4.4}$$

(4.4) with the help of (4.2) implies that

$$\sum_{n=1}^{\infty} (n+p) \, |a_{n+p}| \leq \frac{(p+1)(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} . \tag{4.5}$$

A substitution of (4.5) into (4.3) yields the right-hand inequality.

On the other-hand

$$|f'(z)| \geq r^{p-1}[p-r \sum_{n=1}^{\infty} (p+n) \, |a_{p+n}|]$$

$$\geq pr^{p-1} - \frac{(p+1)(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^p.$$

This completes the proof.

COROLLARY 4. If $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} \in C_p(A,B,\alpha)$, then

$$pr^{p-1} - \frac{(B-A)(p-\alpha)(p+1)^2}{p[1+B+(B-A)(p-\alpha)]} r^p < |f'(z)| <$$

$$pr^{p-1} + \frac{(B-A)(p-\alpha)(p+1)^2}{p[1+B+(B-A)(p-\alpha)]} r^p \quad (|z| = r).$$

Equality holds for $f(z)=z^p - \frac{(B-A)(p-\alpha)(p+1)}{p[1+B+(B-A)(p-\alpha)]} z^{p+1} (z= \pm r)$.

5. RADIUS OF CONVEXITY FOR THE CLASS $T_p^*(A,B,\alpha)$.

THEOREM 6. If $f(z) \in T_p^*(A,B,\alpha)$, then $f(z)$ is p-valently convex in the disc

$$|z| < R_p = \inf_n \left[\frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)} \left(\frac{p}{n+p}\right)^2\right]^{\frac{1}{n}} \quad (n=1,2,\ldots). \tag{5.1}$$

The result is sharp, with the extremal function

$$f(z) = z^p - \frac{(B-A)\ (P-\alpha)}{(1+B)\ n + (B-A)(p-\alpha)} z^{p+n}.$$

PROOF. It is sufficient to show that $\left|\left(1+\frac{zf''(z)}{f'(z)}\right) - p\right| < p$ for $|z| < R_p$.

We have

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right) - p\right| = \left| \frac{-\sum_{n=1}^{\infty} n(n+p)\ |a_{n+p}| z^n}{p - \sum_{n=1}^{\infty} (n+p)\ |a_{n+p}| |z|^n} \right|$$

$$< \frac{\sum_{n=1}^{\infty} n(n+p)\ |a_{n+p}|\ |z|^n}{p - \sum_{n=1}^{\infty} (n+p)\ |a_{n+p}| |z|^n}.$$

Thus

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right) - p\right| < p \text{ if}$$

$$\sum_{n=1}^{\infty} (n+p)^2 |a_{n+p}|\ |z|^n < p^2$$

or

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^2 |a_{n+p}|\ |z|^n < 1. \tag{5.2}$$

According to <u>Theorem 1</u>, $\sum_{n=1}^{\infty} \frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)} |a_{n+p}| < 1$.

Hence (5.2) will be true if

$$\left(\frac{n+p}{p}\right)^2 |z|^n < \frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)}$$

or if

$$|z| < \left[\frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)} \cdot \left(\frac{p}{n+p}\right)^2\right]^{\frac{1}{n}} \quad (n=1,2,\ldots). \tag{5.3}$$

The theorem follows easily from (5.3).

6. CLOSURE THEOREMS.

In this section we shall prove that the classes $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$ are closed under convex linear combinations.

THEOREM 7. If $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$ and $g(z) = z^p - \sum_{n=1}^{\infty} |b_{n+p}| z^{n+p}$ are in $T_p^*(A,B,\alpha)$, then $h(z) = z^p - \frac{1}{2} \sum_{n=1}^{\infty} |a_{n+p} + b_{n+p}| z^{n+p}$ is also in $T_p^*(A,B,\alpha)$.

PROOF. Since $f(z)$ and $g(z)$ are in $T_p^*(A,B,\alpha)$, we have

$$\sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-\alpha)]|a_{n+p}| < (B-A)(p-\alpha) \tag{6.1}$$

and

$$\sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-\alpha)]|b_{n+p}| < (B-A)(p-\alpha). \tag{6.2}$$

From (6.1) and (6.2) we get

$$\tfrac{1}{2} \sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-\alpha)] \, |a_{n+p}+b_{n+p}| < (B-A)(p-\alpha)$$

which implies that $h(z) \in T_p^*(A,B,\alpha)$.

The following theorem can be proven similarly.

THEOREM 8. $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$ and $g(z)=z^p - \sum_{n=1}^{\infty} |b_{n+p}| z^{n+p}$ are in $C_p(A,B,\alpha)$, then $h(z) = z^p - \frac{1}{2} \sum_{n=1}^{\infty} |a_{n+p}+b_{n+p}| z^{n+p}$ is also in $C_p(A.B.\alpha)$.

THEOREM 9. Let $f_p(z) = z^p$, $f_{n+p}(z) = z^p - \frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)} z^{n+p}$ $(n=1,2,3,\ldots)$.

Then $f \in T_p^*(A,B,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=o}^{\infty} \lambda_{n+p} f_{n+p}(z) \text{ where } \lambda_{n+p} > o \text{ and } \sum_{n=o}^{\infty} \lambda_{n+p} = 1.$$

PROOF. Suppose $f(z) = \sum_{n=o}^{\infty} \lambda_{n+p} f_{n+p}(z) = z^p \sum_{n=o}^{\infty} \frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)} \lambda_{n+p} z^{n+p}$.

Then

$$\sum_{n=1}^{\infty} [\lambda_{n+p} \frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)} \cdot (\frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)})]$$

$$= \sum_{n=1}^{\infty} \lambda_{n+p} \leq 1 - \lambda_p \leq 1.$$

So by Theorem 1, $f(z) \in T_p^*(A,B,\alpha)$.

Conversely suppose $f(z) \in T_p^*(A,B,\alpha)$. Then

$$|a_{n+p}| \leq \frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)} .$$

Setting $\lambda_{n+p} = \frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)} |a_{n+p}|$ $(n=1,2,\ldots)$,

and

$$\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_{n+p},$$

we have

$$f(z) = \sum_{n=o}^{\infty} \lambda_{n+p} \; f_{n+p}(z).$$

This completes the proof of the theorem.

REMARKS. (1) Putting $\alpha=o$ in the above theorems we get the results obtained by R.M. Goel and N.S. Sohi [1].

(2) Putting p=1 and taking $A= -\beta$, $B=\beta$, where $o < \beta \leq 1$, in the above theorems we get the results obtained by Gupta and Jain [2].

(3) Putting p=1 and taking A= -1, B=1 in the above theorems we get the results obtained by Silverman [3].

ACKNOWLEDGMENT. In conclusion, I would like to thank Professor Dr. H. Silverman for reading the manuscript and for helpful suggestions.

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