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ON A CLASS OF POLYNOMIALS ASSOCIATED WITH THE CLIQUES IN A GRAPH AND ITS APPLICATIONS

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ABSTRACT. The clique polynomial of a graph is defined. An explicit formula is then derived for the clique polynomial of the complete graph. A fundamental theorem and a reduction process is then given for clique polynomials. Basic properties of the polynomial are also given. It is shown that the number theoretic functions defined by Menon are related to clique polynomials. This establishes a connection between the clique polynomial and decompositions of finite sets, symmetric groups and analysis.

KEYWORDS AND PHRASES. Clique, Clique Polynomial, Decomposition, Generating Function, Cover, Weights, Complete Graph, F-polynomial, Incorporated Graph, Proper Complete Graph, Proper Cover, Matching Polynomial.
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1. INTRODUCTION.

The graphs considered here are finite and without loops or multiple edges. Let G be such a graph. We define a *clique* in G to be a subgraph of G which is a complete graph. It is not necessarily maximal. A *clique cover* (or simply, a *cover*) of G is a spanning subgraph of G whose components are all cliques, i.e., a node disjoint set of cliques which cover all the nodes of G. With every clique α in G, let us associate an indeterminate or *weight* w_α, and with every cover C, a weight

$$w(C) = \Pi\, w_\alpha ,$$

where the product is taken over all the elements of C. Then the *clique polynomial* of G is

$$\Sigma\, w(C) ,$$

where the summation is taken over all the covers of G.

Let us give each clique with n nodes a weight w_n. Then the (*general*) clique polynomial is denoted by $K(G\,;\underline{w})$, where $\underline{w} = (w_1, w_2, w_3, \ldots)$ is a *general weight vector*. This polynomial is a polynomial in the indeterminates w_1, w_2, etc. If we put $w_n = w$, for all values of n (i.e., $\underline{w} = (w, w, w, \ldots)$), then the resulting polynomial is called the *simple clique polynomial* of G. This polynomial is denoted by $K(G\,;w)$.

We derive explicit formulae for the general and simple clique polynomials of the complete graph and for their generating functions. A reduction algorithm is then given for clique polynomials. We show that a number-theoretic function defined by Menon [3] is really the clique polynomial of a complete graph, when weights are assigned in a particular manner. Finally, we establish some interesting connections between the clique polynomial and various combinatorial results. Central to this section of the article is Menon's paper [3].

In the material which follows, we denote the generating function for $K(G;\underline{w})$, (with indicator function t) by $K(G;\underline{w},t)$. Lower limits of summations are zero unless otherwise defined. When unspecified, upper limits are infinity, or the largest value of the variable which makes sense in the context of the summand. For brevity, we write $k(G)$ for $K(G;\underline{w})$.

2. *CLIQUE POLYNOMIALS OF COMPLETE GRAPHS.*

If, instead of cliques, we take the components of a cover of G to be members of a general family F of connected graphs, then the resulting polynomial is called the *F-polynomial* of G. We will not give an independent derivation of the generating function for the clique polynomial of the complete graph. Instead, we use a result given in Farrell [1] (Theorem 1) on the generating function for the general F-polynomial of the complete graph.

LEMMA 1.

The generating function for the F-polynomial of the complete graph is

$$F(K_p;\underline{w},t) = \sum_p \frac{F(K_p;\underline{w})t^p}{p!} = \exp\Big[\sum_{i=1}^{p}\Big(\frac{\phi_i w_i t^i}{i!}\Big)\Big],$$

where ϕ_i is the number of spanning subgraphs of K_i which are members of the family F.

In the case of the clique polynomial, ϕ_i is the number of spanning subgraphs of K_i which are complete graphs. Thus $\phi_i = 1$, for all i. Hence we get the following result.

THEOREM 1.

$$K(K_p;\underline{w},t) = \exp\Big[\sum_{i=1} \frac{w_i t^i}{i!}\Big].$$

The following corollary is obtained by extracting the coefficient of $\frac{t^p}{p!}$.

COROLLARY 1.1.

$$K(K_p;\underline{w}) = p!\,\sum\Big[\prod_{i=1}^{p} \frac{1}{j_i!}\Big(\frac{w_i}{i!}\Big)^{j_i}\Big],$$

where the summation is taken over all solutions of $\sum_i i\,j_i = p$.

We can obtain a recurrence relation for $K(K_p;\underline{w})$ from Theorem 1, using standard techniques (or by direct substitution into Theorem 3 of [3]). This is given in the following theorem, in which the Blissard notation (see Riordan [4]) is used.

THEOREM 2.

$$K^p = w(K+w)^{p-1},$$

$$\text{with}\qquad K^r \equiv K(K_r;\underline{w}) \text{ and } w^r \equiv w_r .$$

Results for the simple clique polynomial of K_p can be obtained from the above results by putting $\underline{w} = (w, w, w, \ldots)$. From Theorem 1, we get

THEOREM 3.

$$K(K_p\,;w,\ t) = \exp[w(\exp t - 1)]\ .$$

This generating function is also the generating function for Stirling numbers of the second kind (see [4], page 32). We denote the k-th Stirling number associated with n, by S(n,k). Hence we have the following interesting result.

THEOREM 4.

$$K(K_p\,;w) = \sum_{k}^{p} S(p,k)w^k\ .$$

The following corollary is immediate.

COROLLARY 4.1.

The number of clique covers with cardinality k in the labelled complete graph K_p is S(p,k).

From Theorem 2 we obtain the following corollary.

COROLLARY 2.1.

$$K^p = w(K + 1)^p\ ,$$

$$\text{where} \qquad K^r \equiv K(K_r\,;w)\ .$$

3. *THE FUNDAMENTAL THEOREM AND ALGORITHM*

Let e be an edge in a graph G. We say that e is *(clique) incorporated* in G if e is required to belong to every clique cover of G as part of a component K_n, for $n > 2$. If a graph G contains an incorporated edge, G is called a *restricted graph*. We normally use an asterisk to denote such a graph. Let H be a subgraph of G. Then *G-H* denotes the graph obtained from G by removing the nodes of H. In order to distinguish between the trivial complete graphs with one and two nodes (i.e. a node and an edge) and the non-trivial ones with more than two nodes, we call K_n a *proper complete graph*, when $n > 2$. A *proper (clique) cover* is a cover in which all components are proper complete graphs.

Let e be an edge of G. We can put the covers in G into three classes (i) those not containing e, (ii) those in which e is a component and (iii) those in which e is part of a proper complete graph. These considerations lead to the following theorem, called the *Fundamental Theorem for clique polynomials.*

THEOREM 5.

Let G be a graph containing an (unincorporated) edge e. Let G′ be the graph obtained from G by deleting e, G″ the graph obtained from G by removing the nodes at the ends of e and G* the graph obtained from G by incorporating e. Then

$$k(G) = k(G') + w_2 k(G'') + k(G^*).$$

An illustration of the Theorem

The diagram is self-explanatory. The internal arrows indicate the edge e. The asterisk indicates the incorporated edge e.

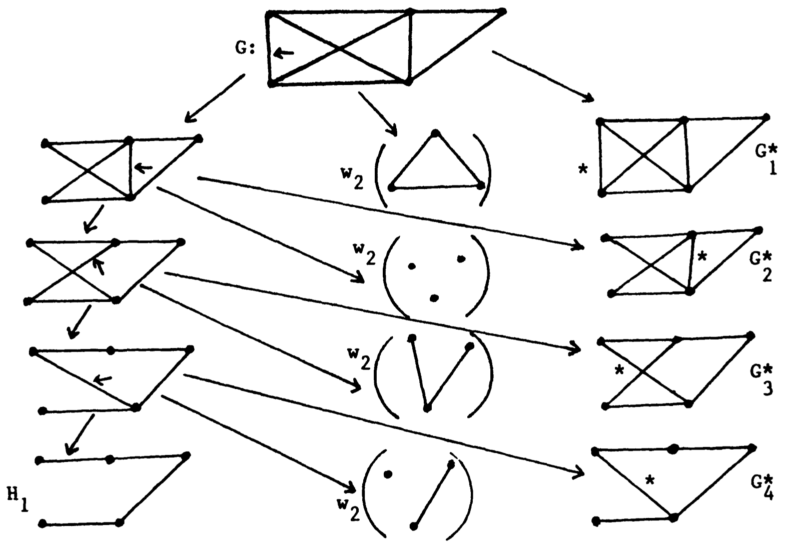

Figure 1

The following corollaries are easily deduced from the definitions and Theorem 5.

COROLLARY 5.1.

If G has no triangles, then $k(G^*) = 0$. Therefore

$$k(G) = k(G') + w_2 k(G'').$$

COROLLARY 5.2.

If e is adjacent to an incorporated edge, then $k(G'') = 0$. Therefore

$$k(G) = k(G') + k(G^*).$$

The following theorem gives a technique for finding clique polynomials of restricted graphs.

THEOREM 6.

Let G^* be a restricted graph containing an incorporated edge e. Then

$$k(G^*) = \Sigma\, w_n(G-H_n),$$

where H_n is a clique of G with n nodes ($n > 2$) and containing the edge e, and the summation is taken over all such clique subgraphs of G.

PROOF.

Let e be part of a clique H_n with $n > 2$ nodes. Then the weight of H_n as a component of clique cover is w_n. The rest of the cover is a cover of the graph $G-H_n$. The result can therefore be deduced.

□

If we restrict the cliques to K_1 and K_2, then the clique polynomial becomes the matching polynomial (see Farrell [2]). The matching polynomial of G is denoted by

$m(G)$. The following result is straightforward. $k_1(G)$ counts proper clique covers.

THEOREM 7.

$$\text{(i)}\quad m(G) = k(G;\ (w_1, w_2, 0, 0, \ldots, 0)).$$

$$\text{(ii)}\quad k_1(G) = k(G;\ (0, 0, w_3, w_4, \ldots, w_p)),\ \text{where } p \text{ is the number of nodes in } G.$$

If G does not have triangles (i.e. G is triangle-free) then the only elements that a cover of G can have are K_1 and K_2. It follows that every clique cover is a matching in G. Hence we have the following result.

THEOREM 8.

If G is a triangle-free graph, then

$$k(G) = m(G)\ .$$

Theorem 5 yields an algorithm for finding clique polynomials of arbitrary graphs. We simply apply the theorem recursively, until we obtain graphs H_i, for which $k(H_i)$ is known. This algorithm is called *the Fundamental Algorithm* for clique polynomials or for brevity, *the reduction process*. We illustrate this algorithm using the graph G in Figure 1. Notice that Corollaries 5.1, 5.2 and Theorem 6 can be very useful when applying the reduction process.

From Figure 1 we get

$$k(G) = m(H_1) + 3w_2(w_1(w_1^{\,2} + w_2)) + w_2k(K_3) + \sum_{i=1}^{4} k(G_i^*),$$

Clearly, $$m(H_1) = w_1^{\,5} + 4w_1^{\,3}w_2 + 3w_1w_2^{\,2}$$

and $$k(K_3) = w_1^{\,3} + 3w_1w_2 + w_3$$

From Theorem 6,

$$k(G_1^*) = w_3(\,|\,) + w_3(\,/\,) + w_4(\cdot)$$

$$= 2w_3(w_1^{\,2} + w_2) + w_1w_4 = 2w_1^{\,2}w_3 + 2w_2w_3 + w_1w_4,$$

Also

$$k(G_2^*) = w_3(\cdot\ \cdot) + w_3(\cdot\ \cdot) + w_3(\cdot\ \cdot) = 3w_1^{\,2}w_3\ .$$

From Corollary 5.1,

$$k(G_3^*) = k(G_4^*) = 0\ .$$

Therefore we get, after simplifying,

$$\underline{k(G) = w_1^{\,5} + 8w_1^{\,3}w_2 + 5w_1^{\,2}w_3 + 9w_1w_2^{\,2} + 3w_2w_3 + w_1w_4}$$

4. *SOME BASIC PROPERTIES OF CLIQUE POLYNOMIALS*

The following theorem is called the *Component Theorem for clique polynomials*. It can be easily proved.

THEOREM 9.

Let G be a graph consisting of r components $G_1, G_2, \ldots, G_r$. Then

$$k(G) = \prod_{i=1}^{r} k(G_i) .$$

Suppose that $w_{r_1} w_{r_2} \dots w_{r_n}$ is a term in k(G) with non-zero coefficient. Then G has a clique cover consisting of the components K_{r_i} $(i = 1, 2, \dots, n)$. Consider the component K_m, where $m = r_i$, for some $0 < i \le n$. This could be replaced by "smaller" clique covers. The weights of these covers will be the terms of $k(K_m)$. Hence we have the following theorem.

THEOREM 10.

If k(G) contains a term in $w_{r_1} w_{r_2} \dots w_{r_n}$ with nonzero coefficient, then every term of the polynomial $w_{r_1} w_{r_2} \dots w_{r_{j-1}} w_{r_{j+1}} \dots w_{r_n} k(K_{r_j})$ also occurs in k(G) with nonzero coefficient, for $j = 1, 2, \dots, n$.

The following theorem characterizes complete graphs in terms of clique polynomials.

THEOREM 11.

Let G be a graph with p nodes. Then G is a complete graph if and only if k(G) contains the term w_p.

5. *CONNECTION WITH MENON'S FUNCTIONS*

Let S be a finite set with cardinality n. Let f be a function defined on all finite sets and depending only on the number of elements in the set; so that f(S) can be written as $f(|S|) = f(n)$. Let D be a partition of S with k elements. We extend the definition of f to all partitions of S by

$$f(D) = \prod_{i=1}^{k} f(S_i) = \prod_{i=1}^{k} f(n_i) ,$$

where S_i is an element of D with n_i elements, and the product is taken over all the k elements in D $(|D| = k)$.

Next, we define the function F on finite sets S by

$$F(S) = \sum_{D} f(D) ,$$

where the summation is taken over all the partitions D of S. We refer to the functions f and F as *Menon's functions*. These functions were defined in [3].

A correspondence between the structures of this section and that of Section 1 can be established by letting f be the weight function $f(S) = w_{|S|}$, D be a cover C of some graph G, and

$$f(D) = w(C) .$$

Finally, F(S) corresponds to the clique polynomial of G, i.e.,

$$F(S) = \sum w(C) .$$

Since all the elements are equivalent, all the nodes of the corresponding graph G must be equivalent. Therefore G must by K_n. The following theorem formalizes our discussion.

THEOREM 12.

Let f and F be the two Menon functions defined above and let S be a set with cardinality n. Then

$$\text{(i)} \quad f(k) = w_k$$

and $$\text{(ii)} \quad F(S) = K(K_n ; \underline{w}) ,$$

where $$\underline{w} = (f(1), f(2), f(3), \ldots) .$$

This result can be formally proved by equating the expression for F(n) given in [3] (Equation 2.6) with the expression for $K(K_n ; \underline{w})$ given in Corollary 1.1.

The following corollary is immediate from Theorem 12, by using the generating function given in Theorem 1, and denoting it by G(t). This result is also given in [3].

COROLLARY 12.1.

$$G(t) = \sum_n F(n) \frac{t^n}{n!} = e^{g(t)} ,$$

where $$g(t) = \sum_{i=1} w_i \frac{t^i}{i!} .$$

From Theorem 2, we obtain the following recurrence for F(n), in which the Blissard notation is used.

COROLLARY 12.2.

$$F^n = f(F + f)^{n-1} ,$$

where $$F^r \equiv F(r) \quad \text{and} \quad f^r \equiv f(r) .$$

This corollary is equivalent to the recurrence given in [3] (Equation 4.1), which was obtained by a combinatorial argument.

6. SOME APPLICATIONS.

The clique polynomial can be applied to enumerating problems connected with the decompositions of finite sets. For example, by giving each clique a weight of 1, in the clique polynomial of K_n, we can obtain the total number of partitions of a set with n elements. Also, the coefficient of w^k in the simple clique polynomial of K_n will be the number of partitions of the set into k elements. These results are stated formally in the following theorem.

THEOREM 13.

Let $D_k(n)$ be the number of partitions of a set of n elements into exactly k elements and D(n), the total number of partitions of S. Then

$$\text{(i)} \quad D_k(n) = \text{coefficient of } w^k \text{ in } K(K_n ; w) = S(n,k)$$

and $$\text{(ii)} \quad D(n) = K(K_n ; \underline{w}) ,$$

where $$\underline{w} = (1, 1, 1, \ldots) .$$

Theorem 13 can be easily proved. Alternatively, the result follows from Theorem 1 and Equations 5.4 and 5.5 of [3].

The clique polynomial can also be applied to certain problems in elementary analysis. This is clear from the following theorem, which is also given in [3], in terms of Menon's functions.

THEOREM 14.

Let h be a function of the single variable x, on the complex numbers, and let the

k^{th} derivative $h^{(k)}(x)$ of $h(x)$ exist for $k = 1, 2, \ldots, n$.

Then

$$\frac{d^n}{dx^n}\left(e^{h(x)}\right) = e^{h(x)} K(K_n ; \underline{w}) ,$$

where $\underline{w} = (h^{(1)}(x), h^{(2)}(x), h^{(3)}(x), \ldots)$.

PROOF.

We will use the result given in Theorem 1. Give each clique with i nodes a weight

$$w_i = h^{(i)}(x) .$$

Put

$$g(t) = \sum_{i=1} w_i \frac{t^i}{i!} = \sum_{i=1} h^{(i)}(x) \frac{t^i}{i!}$$

$$= h(x + t) - h(x), \text{ from Taylor's Theorem.}$$

Then from Corollary 12.1, we get

$$G(t) = \exp[g(t)] = \exp[h(x + t) - h(x)] = e^{-h(x)} e^{h(x+t)}$$

The coefficient of $\frac{t^n}{n!}$ in $G(t)$ (using the Taylor expansion of $e^{h(x+t)}$) is

$$e^{-h(x)} \left\{ \frac{d^n}{dx^n} e^{h(x)} \right\} .$$

But $G(t)$ is the generating function for $K(K_n ; \underline{w})$. Therefore

$$K(K_n ; \underline{w}) = e^{-h(x)} \left\{ \frac{d^n}{dx^n} e^{h(x)} \right\} .$$

Hence the result follows.

$\square$

Menon has used the functions in order to obtain various enumeration theorems about symmetric groups. Thus the clique polynomial of K_n can also be applied to these kinds of enumeration problems. This is not surprising, since Theorem 4 of [1] establishes a connection between the F-polynomial of K_n and the cycle index of the symmetric group on n elements.

7. DISCUSSION

The connection between the clique polynomial and Menon's functions is an important one. Menon's functions were abstract number theoretic functions. Now they have been given a tangible combinatorial form. This we hope will contribute to the use of the functions in enumerative problems.

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