EIGENFUNCTION EXPANSION FOR A REGULAR FOURTH ORDER EIGENVALUE PROBLEM WITH EIGENVALUE PARAMETER IN THE BOUNDARY CONDITIONS

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1. INTRODUCTION.

The regular right-definite eigenvalue problems for second order differential equations with eigenvalue parameter in the boundary conditions, have been studied in Walter [1], Fulton [2] and Hinton [3].

The object of this paper is to prove the expansion theorem for the following regular fourth order eigenvalue problem:

$$\left. \begin{aligned} u: \quad &= (Ku'')'' - (Pu')' + qu = \lambda u, \quad x \in [a, b] \\ u(a) &= (Pu')(a) = (Ku'')(a) = 0 \\ (Ku''')(b) &- (Pu')(b) = -\lambda u(b) \end{aligned} \right\} \quad (1.1)$$

where P, q and K are continuous real-valued functions on $[a, b]$. We assume that $P(x) > 0$, $q(x) > 0$, and $K(x) > 0$ while λ is a complex number.

Recently, Zayed [4] has studied the special case of the problem (1.1) wherein $K(x) = \alpha^2$, α^2 is a constant and $q(x) = 0$.

Further, problem (1.1), in general, describes the transverse motion of a rotating beam with tip mass, such as a helicopter blade (Ahn [5]) or a bob pendulum suspended from a wire (Ahn [6]).

Ahn [7] has shown that the set of eigenvalues of problem (1.1) is not empty, has no finite accumulation points and is bounded from below. He used an integral-equation approach.

In this paper, our approach is to give a Hilbert space formulation to the problem (1.1) and self-adjoint operator defined in it such that (1.1) can be considered as the eigenvalue problem of this operator.

2. HILBERT SPACE FORMULATION.

We define a Hilbert space H of two-component vectors by

$$H = L^2(a, b) \oplus \mathbb{C};$$

with inner product

$$\langle f, g \rangle = \int_a^b f_1 \bar{g}_1 dx + f_2 \bar{g}_2, \quad f, g \in H \quad (2.1)$$

and norm

$$\|f\|_H^2 = \int_a^b |f_1|^2 dx + |f_2|^2 \quad (2.2)$$

where

$$f = (f_1, f_2) = (f_1(x), f_1(b)) \in H$$

and

$$g = (g_1, g_2) = (g_1(x), g_1(b)) \in H.$$

We can define a linear operator $A: D(A) \rightarrow H$ by

$$Af = (\tau f_1, -(Kf_1''')(b) + (Pf_1')(b)) \quad \forall f = (f_1, f_2) \in D(A) \quad (2.3)$$

where the domain $D(A)$ of A is a set of all $f = (f_1, f_2) \in H$ which satisfy the following:

(i) f_1, f_1', f_1'' and f_1''' are absolutely continuous with

$$\tau f_1 \in L^2(a, b) \text{ and } \int_a^b (K|f_1''|^2 + P|f_1'|^2 + q|f_1|^2) dx < \infty.$$

(ii) $f_1(a) = (Pf_1')(a) = (Kf_1'')(a) = 0$

(iii) $f_2 = f_1(b).$

REMARK 2.1. The parameter λ is an eigenvalue of (1.1) and f_1 is a corresponding eigenfunction of (1.1) if and only if

$$f = (f_1, f_1(b)) \in D(A) \quad \text{and} \quad Af = \lambda f \quad (2.4)$$

Therefore, the eigenvalues and the eigenfunctions of problem (1.1) are equivalent to the eigenvalues and the eigenfunctions of operator A .

We consider the following assumptions:

$$\begin{aligned} (i) \quad \lim_{x \rightarrow b} [K'(x)f_1(x) - K(x)f_1'(x)] &= 0, \\ (ii) \quad \lim_{x \rightarrow b} [K'(x)\bar{g}_1(x) - K(x)\bar{g}_1'(x)] &= 0. \end{aligned} \quad (2.5)$$

LEMMA 2.1. The linear operator A in H is symmetric.

PROOF. On using the boundary conditions of (1.1) we get,

$$\begin{aligned} \langle Af, g \rangle &= \int_a^b (\tau f) \bar{g}_1 dx + [-(Kf_1'')(b) + (Pf_1')(b)] \bar{g}_1(b) \\ &= \int_a^b (Kf_1'') \bar{g}_1 dx - \int_a^b (Pf_1')' \bar{g}_1 dx + \int_a^b q f_1 \bar{g}_1 dx - (Kf_1''')(b) \bar{g}_1(b) \\ &\quad + (Pf_1')(b) \bar{g}_1(b) \end{aligned} \quad (2.6)$$

Integrating the first term of (2.6) by parts four times and integrating the second term of (2.6) by parts twice, we get

$$\begin{aligned} \langle Af, g \rangle &= \int_a^b f_1 [(\bar{K}g_1'')'' - (\bar{P}g_1')' + \bar{q}g_1] dx + f_1(b) [-(\bar{K}g_1''')(b) + (\bar{P}g_1')(b)] \\ &\quad + f_1''(b) [K'(b)\bar{g}_1(b) - K(b)\bar{g}_1'(b)] - \bar{g}_1''(b) [K'(b)f_1(b) - K(b)f_1'(b)] \end{aligned}$$

Applying the conditions (2.5) and using the boundary conditions of (1.1), we obtain

$$\langle Af, g \rangle = \int_a^b f_1 (\tau \bar{g}_1) dx + f_1(b) [-(\bar{K}g_1''')(b) + (\bar{P}g_1')(b)] = \langle f, Ag \rangle.$$

REMARK. 2.2. For all $f = (f_1, f_2)$ in $D(A)$ and $f_2 = f_1(b) \neq 0$, the domain $D(A)$ is dense in H .

Since the operator A in H is symmetric and dense in H , A is self-adjoint.

3. THE BOUNDEDNESS.

We shall show that the self-adjoint operator A is unbounded from above and bounded from below. We also show that A is strictly positive.

LEMMA 3.1.

- (i) If f, f' are absolutely continuous with $f(a) = 0$ and $P(x) > 0$ in $[a, b]$, then we have $P(x) > c_1$ for some constant $c_1 > 0$ such that

$$\int_a^b P(x) |f'(x)|^2 dx > c_1 |f(b)|^2.$$

- (ii) For $f \in C^2[a, b]$, there exists a positive constant c_2 such that

$$\int_a^b |f(x)|^2 dx \leq c_2 \int_a^b |f''(x)|^2 dx$$

PROOF.

(i) Since $P(x) > 0$ in $[a, b]$, we have $P(x) > c_1$ for some $c_1 > 0$.

Consequently, on using Schwartz's inequality, we get

$$\int_a^b P(x) |f'(x)|^2 dx > c_1 \int_a^b |f'(x)|^2 dx > c_1 \left[\int_a^b |f'(x)| dx \right]^2 > c_1 |f(b)|^2$$

where $\int_a^b f'(x) dx = f(b) - f(a) = f(b)$, Since $f(a) = 0$.

(ii) By using Theorem 2 in [8, p.67], we have for $f(x) \in C^1[a, b]$,

$$\int_a^b |f(x)|^2 dx < 4(b-a)^2 \int_a^b \left| \frac{d|f(x)|}{dx} \right|^2 dx$$

Since $\left| \frac{d|f(x)|}{dx} \right|^2 < 4 \left| \frac{df(x)}{dx} \right|^2$,

then

$$\int_a^b |f(x)|^2 dx < 4(b-a)^2 \int_a^b \left| \frac{d|f(x)|}{dx} \right|^2 dx < 16(b-a)^2 \int_a^b |f'(x)|^2 dx \quad (3.1)$$

Applying (3.1) again for $|f'(x)|$, we get

$$\int_a^b |f'(x)|^2 dx < 16(b-a)^2 \int_a^b |f''(x)|^2 dx \quad (3.2)$$

from (3.1) and (3.2) we get

$$\int_a^b |f(x)|^2 dx < c_2 \int_a^b |f''(x)|^2 dx \text{ where the constant } c_2 = 256(b-a)^4.$$

LEMMA 3.2. The linear operator A is bounded from below.

PROOF. On using the boundary conditions of (1.1) we get

$$\begin{aligned} \langle Af, f \rangle &= \int_a^b (\tau f_1) \bar{f}_1 dx + [-(Kf_1'')(b) + (Pf_1')(b)] \bar{f}_1(b) \\ &= \int_a^b (Kf_1')'' \bar{f}_1 dx - \int_a^b (Pf_1')' \bar{f}_1 dx + \int_a^b q f_1 \bar{f}_1 dx - (Kf_1'')(b) \bar{f}_1(b) \\ &\quad + (Pf_1')(b) \bar{f}_1(b). \end{aligned} \quad (3.3)$$

Integrating (3.3) by parts twice and using the boundary conditions of (1.1), we obtain

$$\begin{aligned} \langle Af, f \rangle &= f_1''(b) [K'(b) \bar{f}_1(b) - K(b) \bar{f}_1'(b)] + \int_a^b K |f_1'|^2 dx \\ &\quad + \int_a^b P |f_1'|^2 dx + \int_a^b q |f_1|^2 dx. \end{aligned}$$

On using (2.5) (ii) and lemma (3.1), we get

$$\begin{aligned} \langle Af, f \rangle &> \int_a^b \frac{K(x)}{c_2} |f_1(x)|^2 dx + c_1 |f_1(b)|^2 + \int_a^b q(x) |f_1(x)|^2 dx \\ &= \int_a^b \left[\frac{K(x)}{c_2} + q(x) \right] |f_1(x)|^2 dx + c_1 |f_2|^2 \end{aligned}$$

$$> c_3 \int_a^b |f_1(x)|^2 dx + c_1 |f_2|^2$$

where

$$c_3 = \inf_{x \in [a,b]} \left[\frac{K(x)}{c_2} + q(x) \right]$$

Therefore

$$\langle Af, f \rangle > c \|f\|^2 \quad (3.4)$$

where the constant $c = \min(c_3, c_1)$.

It follows, from (3.4), that the operator A is bounded from below. Since $c_1 > 0$, $K(x) > 0$, $q(x) > 0$, $c_2 > 0$ and $c = \min(c_3, c_1)$ then the constant c is positive ($c > 0$) and hence A is strictly positive.

REMARK 3.1.

- (i) Since A is a symmetric operator (from lemma 2.1) then A has only real eigenvalues.
- (ii) By Lemma 3.2, we deduce that the set of all eigenvalues of A is also bounded from below.
- (iii) Since A is strictly positive, then the zero is not an eigenvalue of A .

By using theorem 3 in [8, p.60] we can state that:

Since A in H is symmetric and bounded from below, then for every eigenvalue λ_i of A in H , $\lambda_i > c$ where the constant c is the same as in (3.4). This means that $0 < c < \lambda_1 < \lambda_2 < \dots < \lambda_i$ according to the size and $\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$. This implies that the set of all eigenvalues of A is unbounded from above.

REMARK 3.2. Since the operator A is self-adjoint, then A has only real eigenvalues and the eigenfunctions of A are orthonormal. By using theorem 3 in [8, p.30], the density of the domain $D(A)$ in H gives us the completeness of the orthonormal system of eigenfunctions $\phi_1, \phi_2, \phi_3, \dots$ of A .

4. THE EIGENFUNCTIONS OF THE OPERATOR A .

We suppose $\phi_\lambda(x)$, $\psi_\lambda(x)$, $\chi_\lambda(x)$ and $\gamma_\lambda(x)$, where $\lambda \in \mathbb{C}$ is not an eigenvalue of A , are the fundamental set of solutions of the fourth order differential equation of (1.1) with the initial conditions:

$$\phi_\lambda(a) = 0, \quad (P\phi'_\lambda)(a) = 0, \quad \phi''_\lambda(a) = 1, \quad (K\phi'''_\lambda)(a) = 0 \quad (4.1)$$

$$\psi_\lambda(a) = 0, \quad (P\psi'_\lambda)(a) = 0, \quad \psi''_\lambda(a) = 0, \quad (K\psi'''_\lambda)(a) = 1 \quad (4.2)$$

$$\chi_\lambda(b) = 0, \quad (P\chi'_\lambda)(b) = 1, \quad \chi''_\lambda(b) = 0, \quad (K\chi'''_\lambda)(b) = 1 \quad (4.3)$$

$$\gamma_\lambda(b) = 1, \quad (P\gamma'_\lambda)(b) = 1+\lambda, \quad \gamma''_\lambda(b) = 0, \quad (K\gamma'''_\lambda)(b) = 1 \quad (4.4)$$

Therefore the Wronskian is

$$W = \lim_{x \rightarrow b} [\chi_\lambda(x)(P\gamma'_\lambda)(x) - (P\chi'_\lambda)(x)\gamma_\lambda(x)] = -1 \neq 0$$

Thus the solutions $\phi_\lambda(x)$, $\psi_\lambda(x)$, $\chi_\lambda(x)$ and $\gamma_\lambda(x)$ are linearly independent of $\tau u = \lambda u$. Putting $x = b$, we obtain the Wronskian in the form:

$$\begin{aligned} W &= \psi_\lambda''(b) [\lambda \phi_\lambda(b) - (P\phi_\lambda')(b) + (K\phi_\lambda''')(b)] \\ &\quad - \phi_\lambda''(b) [\lambda \psi_\lambda(b) - (P\psi_\lambda')(b) + (K\psi_\lambda''')(b)] \neq 0 \end{aligned} \quad (4.5)$$

Now, for $f = (f_1, f_2) \in H$, we define $\phi = (\phi_1, \phi_2) \in D(A)$ as the unique solution of $(\lambda I - A)\phi = f$.

Application of variation of parameter method yields the unique solution $\phi \in D(A)$ of $(\lambda I - A)\phi = f$, $f \in H$ with:

$$(\lambda I - \tau) \phi_1 = f_1 \quad (4.6)$$

$$\lambda \phi_1(b) - (P\phi_1')(b) + (K\phi_1''')(b) = f_2$$

Therefore

$$\begin{aligned} \phi_1(x) &= \int_a^b \left[\frac{\phi_\lambda(x)\alpha_1(\tau) + \psi_\lambda(x)\alpha_2(\tau)}{W} \right] f_1(\tau) d\tau \\ &\quad + \int_a^b \left[\frac{\chi_\lambda(x)\alpha_3(\tau) + \gamma_\lambda(x)\alpha_4(\tau)}{W} \right] f_1(\tau) d\tau \\ &\quad + d_1 \phi_\lambda(x) + d_2 \psi_\lambda(x) + d_3 \chi_\lambda(x) + d_4 \gamma_\lambda(x), \end{aligned} \quad (4.7)$$

where

$$\alpha_1(\tau) = \frac{-P(\tau)}{K(\tau)} \begin{vmatrix} \psi_\lambda(\tau) & \chi_\lambda(\tau) & \gamma_\lambda(\tau) \\ \psi_\lambda'(\tau) & \chi_\lambda'(\tau) & \gamma_\lambda'(\tau) \\ \psi_\lambda''(\tau) & \chi_\lambda''(\tau) & \gamma_\lambda''(\tau) \end{vmatrix}$$

$$\alpha_2(\tau) = \frac{P(\tau)}{K(\tau)} \begin{vmatrix} \phi_\lambda(\tau) & \chi_\lambda(\tau) & \gamma_\lambda(\tau) \\ \phi_\lambda'(\tau) & \chi_\lambda'(\tau) & \gamma_\lambda'(\tau) \\ \phi_\lambda''(\tau) & \chi_\lambda''(\tau) & \gamma_\lambda''(\tau) \end{vmatrix}$$

$$\alpha_3(\tau) = \frac{-P(\tau)}{K(\tau)} \begin{vmatrix} \phi_\lambda(\tau) & \psi_\lambda(\tau) & \gamma_\lambda(\tau) \\ \phi_\lambda'(\tau) & \psi_\lambda'(\tau) & \gamma_\lambda'(\tau) \\ \phi_\lambda''(\tau) & \psi_\lambda''(\tau) & \gamma_\lambda''(\tau) \end{vmatrix}$$

and

$$\alpha_4(\tau) = \frac{P(\tau)}{K(\tau)} \begin{vmatrix} \phi_\lambda(\tau) & \psi_\lambda(\tau) & \chi_\lambda(\tau) \\ \phi_\lambda'(\tau) & \psi_\lambda'(\tau) & \chi_\lambda'(\tau) \\ \phi_\lambda''(\tau) & \psi_\lambda''(\tau) & \chi_\lambda''(\tau) \end{vmatrix}$$

while d_1 , d_2 , d_3 and d_4 are constants.

Calculation of $\phi_1(b)$, $\phi_1'(b)$ and $\phi_1'''(b)$ from (4.7) and substitution into (4.6) with the initial conditions (4.3) and (4.4), we can get the constants d_1 , d_2 , d_3 and d_4 as follows:

$$d_1 = \frac{1}{W} \left[-f_2 \psi_\lambda''(b) + \int_a^b \alpha_1(t) f_1(t) dt \right],$$

$$d_2 = \frac{1}{W} \left[f_2 \phi_\lambda''(b) + \int_a^b \alpha_2(t) f_1(t) dt \right]$$

and $d_3 = d_4 = 0$.

Consequently, we deduce that

$$\phi_1(x) = \frac{f_2}{W} [\psi_\lambda(x) \phi_\lambda''(b) - \phi_\lambda(x) \psi_\lambda''(b)] + \int_a^b G(x, t, \lambda) f_1(t) dt \quad (4.8)$$

and

$$\phi_2 = \phi_1(b)$$

where $G(x, t, \lambda)$ is the Green's function defined by:

$$G(x, t, \lambda) = \left. \begin{array}{ll} \frac{\phi_\lambda(x) \alpha_1(t) + \psi_\lambda(x) \alpha_2(t)}{W} & a < x < t < b \\ \frac{\chi_\lambda(x) \alpha_3(t) + \gamma_\lambda(x) \alpha_4(t)}{W} & a < t < x < b \end{array} \right\} \quad (4.9)$$

The form of equations (4.8) and (4.9) shows that the inverse operator $(\lambda I - A)^{-1}$ is actually compact; for details of argument of theorem 5 in [8, p.120] can be used.

5. EXPANSION THEOREM.

We now arrive at the problem of expanding an arbitrary function $f(x) \in H$ for $x \in [a, b]$ in terms of the eigenfunctions of (1.1). The results of our investigations are summarized in the following theorem:

THEOREM 5.1. The operator A in H has unbounded set of real eigenvalues of finite multiplicity, (they have at most multiplicity four), without accumulation points in $(-\infty, \infty)$ and they can be ordered according to the size, $0 < c < \lambda_1 < \lambda_2 < \dots < \lambda_i$ with $\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$. If the corresponding eigenfunctions $\phi_1, \phi_2, \phi_3, \dots$ form a complete orthonormal system, then for any function $f(x) \in H$, we have the expansion:

$$f(x) = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle \phi_i \quad (4.10)$$

which is a uniformly convergent series.

The above theorem has some interesting corollaries for particular choices of f .

COROLLARY 4.1. If $f_1 \in L^2(a, b)$ and $f = (f_1, 0) \in H$, then we have

$$(i) \quad f_1 = \sum_{i=1}^{\infty} \left(\int_a^b f_1 \phi_{i1} dx \right) \phi_{i1}(x)$$

$$(ii) \quad 0 = \sum_{i=1}^{\infty} \left(\int_a^b f_1 \phi_{i1} dx \right) \phi_{i2}$$

COROLLARY 4.2. If $\phi_i = (\phi_{i1}(x), \phi_{i2}) \in D(A)$ and $f = (0, 1) \in H$, we have:

$$(i) \quad 0 = \sum_{i=1}^{\infty} \phi_{i2} \phi_{i1}(x) = \sum_{i=1}^{\infty} \phi_{i1}(b) \phi_{i1}(x).$$

$$(ii) \quad 1 = \sum_{i=1}^{\infty} [\phi_{i2}]^2 = \sum_{i=1}^{\infty} [\phi_{i1}(b)]^2.$$

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