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THE DIOPHANTINE EQUATION $n^i + 1 = k(dn - 1)$

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ABSTRACT. The Diophantine equation of the title is solved for $i = 3,4$ and an infinite family of solutions were found for $i \geqq 5$.

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1. INTRODUCTION.

In this note we will find an infinite family of solutions of

$$n^i + 1 = k(dn - 1),\ k,\ d > 0, \tag{1.1}$$

and for $i = 3,4$, all solutions will be obtained.

This equation, for $i = 3$, was a problem in [1] and solved by Ligh [2]. The solutions were $n = 1,2,3$ and 5. For $i = 4$, the problem was proposed in [3] by K. Wilke and solved by the proposer in [4]. He showed that there are infinitely many values of n and all can be obtained from a single recurrence relation. It is the purpose of this note to show that there are infinite solutions for $i \geqq 5$ and, unlike $i = 4$, no single recurrence relation will yield all solutions.

2. SOLUTIONS FOR ARBITRARY i.

Reducing equation (1.1) to a congruence modulo n yields

$$k \equiv -1 \pmod{n}.$$

Hence there is a positive integer r such that $k = (rn - 1)$ and (1.1) can be written as

$$n^i + 1 = (dn - 1)(rn - 1). \tag{2.1}$$

We wish to find the set S of all triples (d,n,r) which satisfy (2.1). Clearly, if (d,n,r) is in S, then so is (r,n,d).

Finding an infinite family of solutions of (2.1) is facilitated by the following result:

THEOREM 1. If (d,n,r) satisfies (2.1) then there are positive integers x and y such that (x,d,n) and (n,r,y) also satisfy (2.1).

PROOF. Multiplying (2.1) by d^{i-1} and dividing by dn - 1 yields the following:

$$n^{i-1}d^{i-2} + n^{i-2}d^{i-3} + \dots + n + \frac{n + d^{i-1}}{dn - 1} = d^{i-1}(rn - 1), \tag{2.2}$$

Since the right side of (2.2) is an integer it follows that dn - 1 must divide $n + d^{i-1}$. Let x be the positive integer such that

$$\frac{n + d^{i-1}}{dn - 1} = x. \tag{2.3}$$

Now multiplying (2.3) by d and rearranging gives the following equation:

$$d^i + 1 = (xd - 1)(nd - 1). \tag{2.4}$$

Hence (x,d,n) satisfies (2.1) and similarly there is a positive integer y such that (n,r,y) also satisfies (2.1).

Now equation (2.1) can be rewritten as follows.

$$n_1^i + 1 = (n_0 n_1 - 1)(n_2 n_1 - 1), \tag{2.5}$$

where (n_0,n_1,n_2) satisfies (2.1). Thus for each positive integer j, according to Theorem 1, there are integers n_{j-1}, n_j and n_{j+1} such that (n_{j-1},n_j,n_{j+1}) satisfies (2.1) and

$$n_j^i + 1 = (n_{j-1}n_j - 1)(n_{j+1}n_j - 1). \tag{2.6}$$

THEOREM 2. If $i > 3$, then (2.1) has an infinite family of solutions.

PROOF. For $n = 1$, (2,1,3) and (3,1,2) are the only triples satisfying (2.1). Starting with either one, we obtain an infinite family of solutions if for each j, $n_{j+1} > n_j$. Clearly $n_1 = 1$ and $n_2 > n_1$. Suppose $n_j > n_{j-1}$, solving for n_{j+1} in (2.6), we have

$$n_{j+1} = \frac{n_j^{i-1} + n_{j-1}}{n_j n_{j-1} - 1} > \frac{n_j^{i-1}}{n_j^2} = n_j^{i-3}.$$

Hence, by induction, $n_{j+1} > n_j$ if $i > 3$ and (2.1) has an infinite family of solutions:

$(2,1,3) \rightarrow \dots \rightarrow (n_{j-1},n_j,n_{j+1}) \rightarrow \dots$, or $(3,1,2) \rightarrow \dots \rightarrow (n_{j-1},n_j,n_{j+1}) \rightarrow \dots$.

3. THE CASE $i = 3$ OR $i = 4$.

For $i = 3$, the only solutions are $n = 1, 2, 3$ and 5 and can be found in [2].

For $i = 4$, the solution were given in [4]. It was shown that if (d,n,r) with $d \leqq r$ is a nontrivial triple (i.e. $n > 1$) satisfying (2.1), then $d < n < r$. Thus, from Theorem 1, there is a positive integer x so that (x,d,n) satisfies (2.1). Moreover, if $d > 1$, then $x < d < n$. By continuing this process, starting from any nontrivial solution of (2.1), eventually one will reach a solution of the form $(1,n,r)$. But when $d = 1$, the solutions of (2.1) with $i = 4$ are $(1,2,9)$ and $(1,3,14)$. Thus the integers n satisfying (2.1) with $i = 4$ are precisely those integers in the sequence $n_1 < n_2 < n_3 <$. where any three consecutive terms satisfy (2.6) with $i = 4$. Furthermore, (2.6) is equivalent to

$$n_{j-1}n_{j+1} = n_j^{i-2} + \frac{n_{j-1} + n_{j+1}}{nj}, \tag{3.1}$$

and thus for any j there is a positive integer k_j such that

$$n_{j-1} + n_{j+1} = k_j n_j. \tag{3.2}$$

It was shown in [4] that if $i = 4$, then $k_j = 5$ for all j. Thus the values of n satisfying (2.1) with $i = 4$ are those in the sequences (with $n_1 = 1$, $n_2 = 2$ or $n_1 = 1$, $n_2 = 3$):

1, 2, 9, 43, 206, 987, ...

and

1, 3, 14, 67, 321, 1538, ...

4. THE CASE $i \geqq 5$.

Even though (2.1) has an infinite family of solutions for $i > 3$, unlike the case $i = 4$, no single recurrence relation will yield all solutions for $i \geqq 5$. We will accomplish this by showing that for each $i \geqq 5$, there is a triple (d,n,r) satisfying (2.1) and yet it cannot be obtained from the two trivial solutions $(3,1,2)$ and $(2,1,3)$. We will call such a triple primitive. Furthermore, we will also prove that, unlike the case $i = 4$, the integer k_j in (3.2) is not a constant. Hence, even though each primitive triple generates an infinite family of solutions, no single recurrence relation, as in the case $i = 4$, can describe the family.

THEOREM 3. For each $i \geqq 5$, there is a primitive triple (d,n,r), $n \geqq 2$, satisfying (2.1) besides the two trivial ones, $(3,1,2)$ and $(2,1,3)$.

PROOF. We will exhibit a triple (d,n,r) satisfying (2.1) that cannot be obtained from either $(3,1,2)$ or $(2,1,3)$ by applying Theorem 1. For i odd or $i = 2^x j$, $x \geqq 1$, j odd and $j \geqq 3$, $n^i + 1$ can be factored. Hence there are integers a and b such that $(2,2,a)$ and $(2^{2^x-1} + 1,2,b)$ satisfy (2.1) for i odd and $i = 2^x j$ respectively. They are primitive because the only triples with 2 in the middle obtained from either $(3,1,2)$ or $(2,1,3)$ are $(1,2,y)$ and $(z,2,1)$.

For $i = 2^x$, there are two cases: (i) if $2^{2^x} + 1$ is not a prime, then $2^{2^x} + 1 = ab$ where a and b are odd. Thus $(d,2,r)$ is primitive where $d = \frac{a+1}{2} > 1$ and $r = \frac{b+1}{2} > 1$. (ii) if $2^{2^x} + 1$ is a prime, then the only known values of x are $x = 0,1,2,3,4$. We need to consider $x = 3$ and $x = 4$ only. If $x = 3$, then $i = 8$ and

$$3^8 + 1 = 17 \cdot 386 = (6 \cdot 3 - 1)(129 \cdot 3 - 1).$$

Hence (6,3,129) is primitive because the only triples with 3 in the middle from (3,1,2) or (2,1,3) are (x,3,1) and (1,3,y). If $x = 4$, then $i = 16$ and

$$6^{16} + 1 = 1697 \cdot 1662410081 = (283 \cdot 6 - 1)(277068347 \cdot 6 - 1).$$

Again, starting with (3,1,2) or (2,1,3), applying Theorem 1, one does not get (283,6,277068347). Hence it is primitive.

Recall that when $i = 4$, there are only two primitive triples satisfying (2.1), namely (3,1,2) and (2,1,3). Starting from either one, using Theorem 1, one obtains an infinite family of solutions and furthermore, all of them are given by the recurrence relation (3.1) with $k_j = 5$ for all j.

In order to show $k_j = 5$ for all j when $i = 4$, it was shown in [4] that, according to (3.1) and (3.2),

$$\frac{n_{j-1} + n_{j+1}}{n_j} = \frac{n_j + n_{j+2}}{n_{j+1}} \quad \text{for all } j. \tag{4.1}$$

However, for $i \geq 5$, (4.1) no longer holds.

THEOREM 4. $\dfrac{n_{j-1} + n_{j+1}}{n_j} = \dfrac{n_j + n_{j+2}}{n_{j+1}}$

if and only if $i = 4$.

PROOF. From (2.6),

$$n_{j+2} = \frac{n_{j+1}^{i-1} + n_j}{n_j n_{j+1} - 1} \quad \text{and} \quad n_{j-1} = \frac{n_j^{i-1} + n_{j+1}}{n_{j+1} n_j - 1}.$$

Using the above and simplify, we have

$$\frac{n_j + n_{j+2}}{n_{j+1}} = \frac{n_j^2 + n_{j+1}^{i-2}}{n_{j+1} n_j - 1} \quad \text{and} \quad \frac{n_{j-1} + n_{j+1}}{n_j} = \frac{n_j^{i-2} + n_{j+1}^2}{n_{j+1} n_j - 1}.$$

Hence $n_j^2 + n_{j+1}^{i-2} = n_j^{i-2} + n_{j+1}^2$ implies

$$n_{j+1}^2 (n_{j+1}^{i-4} - 1) = n_j^2 (n_j^{i-4} - 1). \tag{4.2}$$

By Theorems 2, $n_{j+1} > n_j$ for all j and thus (4.2) holds if and only if $i = 4$.

We can conclude from Theorem 4, even though one gets an infinite family of solutions starting with any primitive triple, for $i \geq 5$ no single relation describes all solutions.

5. CONCLUDING REMARKS.

Equation (2.1) can be written as

$$n^{i-1} - drn + (d + r) = 0. \tag{5.1}$$

A similar equation,

$$n^2 + (d + r)n = kdr, \tag{5.2}$$

was investigated by W.R. Utz in [5] and he obtained all solutions. Because of the similarity of (5.1) and (5.2), at least in appearance, and the fact that there are other primitive triples for $i \geq 5$ in (2.1), we conclude this note with the following problems:

Problem 1. Find the solutions of

$$n^i + (d + r)n = kdr.$$

Problem 2. Find all primitive triples, if possible, of equation (2.1) for $i \geq 5$.

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