WEIGHTED ADDITIVE INFORMATION MEASURES

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ABSTRACT. We determine all measurable functions $I, G, L: [0,1] \rightarrow \mathbb{R}$ satisfying the functional equation

$$\sum_{i=1}^n \sum_{j=1}^m I(p_i q_j) = \sum_{i=1}^n \sum_{j=1}^m G(p_i) I(q_j) + \sum_{i=1}^n \sum_{j=1}^m L(q_j) I(p_i)$$

for $P \in \Gamma_n$, $Q \in \Gamma_m$ and for a fixed pair (n, m) , $n \geq 3$, $m \geq 3$, where $G(0) = L(0) = 0$ and $G(1) = L(1) = 1$. This functional equation has interesting applications in information theory.

KEY WORDS AND PHRASES. Weighted additive information measures of sum form type, entropies of degree (α, β) .

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1. INTRODUCTION.

Let $\Gamma_k = \{P = (p_1, \dots, p_k) : p_i \geq 0, \sum_{i=1}^k p_i = 1\}$, $k \geq 2$.

We say that an information measure $I_k : \Gamma_k \rightarrow \mathbb{R}$, $k \geq 2$ is (n, m) -weighted additive ($n, m \in \mathbb{N}$) if there exist weight functions $G_k, L_k : \Gamma_k \rightarrow \mathbb{R}$, $k \geq 2$ such that

$$I_{nm}(P \cdot Q) = G_n(P) I_m(Q) + I_n(P) L_m(Q) \quad , \quad P \in \Gamma_n, Q \in \Gamma_m \quad (1.1)$$

where as usual $P \cdot Q = (p_1 q_1, \dots, p_i q_j, \dots, p_n q_m) \in \Gamma_{nm}$. If in addition I_k, G_k, L_k have the sum property with generating functions $I, G, L : [0,1] \rightarrow \mathbb{R}$, that is

$$I_k(P) = \sum_{i=1}^k I(p_i) \quad , \quad G_k(P) = \sum_{i=1}^k G(p_i) \quad , \quad L_k(P) = \sum_{i=1}^k L(p_i) \quad (1.2)$$

then equation (1.1) goes over into the functional equation

$$\sum_{i=1}^n \sum_{j=1}^m I(p_i q_j) = \sum_{i=1}^n \sum_{j=1}^m G(p_i) I(q_j) + \sum_{i=1}^n \sum_{j=1}^m L(q_j) I(p_i) \quad , \quad (1.3)$$

$P \in \Gamma_n$, $Q \in \Gamma_m$. This functional equation (1.3) is of interest since the special cases

$$G(p) = p \quad , \quad L(p) = p + \lambda I(p) \quad , \quad \lambda \in \mathbb{R} \quad (1.4)$$

and

$$G(p) = p^\alpha \quad , \quad L(p) = p^\beta \quad , \quad \alpha, \beta \in \mathbb{R} \quad (1.5)$$

play important roles in the characterization of the entropies of degree α (Losonczi, [1])

$$I_k^a(P) = \begin{cases} H_k^{(a,1)}(P) = (2^{1-a} - 1)^{-1} \sum_{i=1}^k (p_i^a - p_i) & a \neq 1 \\ H_k^1(P) = - \sum_{i=1}^k p_i \log p_i & a = 1 \end{cases} \quad P \in \Gamma_k \quad (1.6)$$

and degree (α, β) (Sharma and Taneja, [2])

$$I_k^{(\alpha, \beta)}(P) = \begin{cases} H_k^{(\alpha, \beta)}(P) = (2^{1-\alpha} - 2^{1-\beta})^{-1} \sum_{i=1}^k (p_i^\alpha - p_i^\beta) & \alpha \neq \beta \\ H_k^\alpha(P) = - 2^{\alpha-1} \sum_{i=1}^k p_i^\alpha \log p_i & \alpha = \beta \end{cases} \quad P \in \Gamma_k \quad (1.7)$$

respectively. Here we follow the conventions

$$\log = \log_2 \quad , \quad 0 \cdot \log 0 = 0 \quad \text{and} \quad 0^a = 0 \quad , \quad a \in \mathbb{R}. \quad (1.8)$$

The aim of this paper is to determine all measurable triples (I, G, L) satisfying (1.3) for a fixed pair (n, m) , $n \geq 3$, $m \geq 3$ where - because of the known results and the convention (1.8) - we assume

$$G(0) = L(0) = 0 \quad , \quad G(1) = L(1) = 1. \quad (1.9)$$

Thus we determine not only all measurable functions I of (I_k) (see (1.2)) but also all possible choices for G and L in (1.3). Therefore the results due to Kannappan [3-5], Losonczi [1], Sharma and Taneja [2,6] are special cases of our main result. Moreover, if we assume that I is not constant and that G and L are continuous then we can interpret our result in the form that, without loss of generality, we may assume that G and L in (1.3) are continuous, non zero multiplicative functions, that is they are non zero continuous solutions of the functional equation

$$M(p \cdot q) = M(p) \cdot M(q) \quad p, q \in [0, 1]. \quad (1.10)$$

2. MAIN RESULTS.

We make use of the following well known result (Kannappan, [5]).

LEMMA 1. Let $n \geq 3$ be a fixed integer and let $F : [0, 1] \rightarrow \mathbb{R}$ be a measurable function satisfying

$$\sum_{i=1}^n F(p_i) = 0$$

for all $P \in \Gamma_n$. Then there exists a constant a such that

$$F(p) = a(1 - np) \quad , \quad p \in [0,1].$$

Now we are ready to prove our main result which is an extension of the results, mentioned above.

THEOREM 2. Let $I, G, L : [0,1] \rightarrow \mathbb{R}$ be measurable and let I be non constant. Then I, G, L satisfy (1.9) and (1.3) for a fixed pair (n, m) , $n \geq 3$, $m \geq 3$ if, and only if they are of one of the following forms :

$$\begin{aligned} I(p) &= a(p^A - p^B), \\ G(p) &= (1 - b)p^A + bp^B \quad , \quad L(p) = bp^A + (1 - b)p^B \quad , \quad A \neq B \quad , \end{aligned} \quad (2.1)$$

$$\begin{aligned} I(p) &= ap^A \log p \quad , \\ G(p) &= p^A(1 + b \log p) \quad , \quad L(p) = p^A(1 - b \log p) \quad , \quad A \neq 1 \quad , \end{aligned} \quad (2.2)$$

$$I(p) = I(0) + (mn - m - n)I(0)p + dp \log p \quad , \quad G(p) = L(p) = p \quad , \quad (2.3)$$

$$\begin{aligned} I(1) &= 0 \quad , \quad G(1) = 1 \quad , \quad L(1) = 1 \\ I(p) &= ap^A \quad , \quad G(p) = (1 - b)p^A \quad , \quad L(p) = bp^A \quad , \quad p \in [0,1] \end{aligned} \quad (2.4)$$

$$\begin{aligned} I(p) &= ap^A \sin(c \log p) \quad , \quad G(p) = p^A [\cos(c \log p) + b \sin(c \log p)] \quad , \\ L(p) &= p^A [\cos(c \log p) - b \sin(c \log p)] \quad . \end{aligned} \quad (2.5)$$

Here A, B, a, b, c, d are constants and we follow the conventions

$$0^a \cdot \cos(\log 0) = 0 \quad , \quad 0^a \cdot \sin(\log 0) = 0 \quad , \quad a \in \mathbb{R}.$$

PROOF. Obviously, the solutions (I, G, L) given by (2.1) to (2.5) satisfy (1.9) and (1.3). To prove the converse let us introduce the function $I' : [0,1] \rightarrow \mathbb{R}$ defined by

$$I'(p) = I(p) - I(0) - (I(1) - I(0))p \quad , \quad p \in [0,1]. \quad (2.6)$$

It is clear that I' fulfills

$$I'(0) = I'(1) = 0. \quad (2.7)$$

We now show that the triple (I', G, L) also satisfies (1.3). To see this let us put $P = (1, 0, 0, \dots, 0) \in \Gamma_n$ and $Q = (1, 0, 0, \dots, 0) \in \Gamma_m$ into (1.3). Using (1.9) we arrive at

$$I(1) + (nm - 1)I(0) = I(1) + (m - 1)I(0) + I(1) + (n - 1)I(0)$$

or

$$I(1) - I(0) = (mn - m - n)I(0). \quad (2.8)$$

Thus I' can also be written in the form

$$I'(p) = I(p) - I(0) - (mn - m - n)I(0)p. \quad (2.9)$$

Substituting $P \in \Gamma_n$, $Q = (1, 0, 0, \dots, 0) \in \Gamma_m$ and $P = (1, 0, 0, \dots, 0) \in \Gamma_n$, $Q \in \Gamma_m$ separately into (1.3) we get

$$\sum_{i=1}^n G(p_i) (I(1) + (m - 1)I(0)) = (nm - n)I(0) \quad (2.10)$$

and

$$\sum_{j=1}^m L(q_j) (I(1) + (n-1)I(0)) = (nm - m)I(0) \quad (2.11)$$

or, using (2.8)

$$(1 - \sum_{j=1}^n G(p_i)) (nm - n)I(0) = 0 \quad (2.12)$$

and

$$(1 - \sum_{j=1}^m L(q_j)) (nm - m)I(0) = 0, \quad (2.13)$$

respectively.

After these preparations we can see immediately that I', G, L satisfy

$$\sum_{i=1}^n \sum_{j=1}^m I'(p_i q_j) = \sum_{i=1}^n \sum_{j=1}^m G(p_i) I'(q_j) + \sum_{i=1}^n \sum_{j=1}^m L(q_j) I'(p_i) \quad (2.14)$$

for all $P \in \Gamma_n$, $Q \in \Gamma_m$. Putting I' , given by (2.6), into (2.14) and using (1.3) and (2.8), we see that (2.14) is equivalent to

$$(1 - \sum_{i=1}^n G(p_i)) (nm - n)I(0) + (1 - \sum_{j=1}^m L(q_j)) (nm - m)I(0) = 0. \quad (2.15)$$

But (2.15) is indeed valid because of (2.12) and (2.13).

In a further step we derive a functional equation for I' , G and L in which no sums will occur. Setting

$$F(p, q) = I'(p \cdot q) - G(p)I'(q) - L(q)I'(p), \quad p, q \in [0, 1] \quad (2.16)$$

we get from (2.14)

$$\sum_{i=1}^n \sum_{j=1}^m F(p_i, q_j) = 0, \quad P \in \Gamma_n, Q \in \Gamma_m.$$

Since by hypothesis $F : [0, 1]^2 \rightarrow \mathbb{R}$ is measurable in each variable we get from Lemma 1 in Kannappan [3] (This Lemma is an application of the above Lemma 1) that F can be represented in the form

$$\begin{aligned} F(p, q) &= F(p, 0)(1 - mq) + F(0, q)(1 - np) - \\ &\quad - F(0, 0)(1 - mq)(1 - np). \end{aligned} \quad (2.17)$$

Thus (2.16) and (2.17) imply

$$I'(p \cdot q) = G(p)I'(q) + L(q)I'(p), \quad p, q \in [0, 1] \quad (2.18)$$

since (2.17), (1.9) and (2.7) yield $F(p, 0) = F(0, q) = F(0, 0) = 0$. Because of $I'(0) = G(0) = L(0) = 0$ it is enough to solve (2.18) for all $p, q \in (0, 1]$. Complex-valued functional equations of this type were intensively studied by Vincze [7-9]. From these results we get the solutions of (2.18) for $p, q \in (0, 1]$ (Ebanks, [10]) which have one of the following forms :

$$I'(p) = a \cdot M(p) \cdot \log p,$$

$$G(p) = M(p)(1 + b \cdot \log p), \quad L(p) = M(p)(1 - b \cdot \log p), \quad (2.19)$$

$$I'(p) = a(M_1(p) - M_2(p)),$$

$$G(p) = (1 - b)M_1(p) + bM_2(p), \quad L(p) = bM_1(p) + (1 - b)M_2(p), \quad (2.20)$$

$$I'(p) = aM(p)\sin(c \log p), \quad G(p) = M(p)[\cos(c \log p) + b\sin(c \log p)],$$

$$L(p) = M(p)[\cos(c \log p) - b\sin(c \log p)]. \quad (2.21)$$

Here a, b, c are constants and $M, M_1, M_2: (0, 1] \rightarrow \mathbb{R}$ are measurable multiplicative functions. Let us remark that the measurable solutions of (1.10) for $p, q \in (0, 1]$ are either

$$M = 0 \quad \text{or} \quad (2.22)$$

$$M(p) = p^A, \quad A \in \mathbb{R} \quad \text{or} \quad (2.23)$$

$$M(1) = 1, \quad M(p) = 0 \quad \text{for } p \in (0, 1). \quad (2.24)$$

Since I has the form

$$I(p) = I'(p) + I(0) + (nm - n - m)I(0)p, \quad p \in [0, 1] \quad (2.25)$$

(see (2.6) - (2.8)) we can derive the solutions (I, G, L) of (1.9) and (1.3) from (2.19) to (2.24). Let us first consider the case that

$$\sum_{i=1}^n (G(p_i) - p_i) = 0 \quad \text{and} \quad \sum_{j=1}^m (L(q_j) - q_j) = 0 \quad (2.26)$$

for all $P \in \Gamma_n, Q \in \Gamma_m$. Then Lemma 1 implies that

$$G(p) = rp + s, \quad L(p) = r'p + s', \quad p \in [0, 1]$$

where r, r', s, s' are constants. Because of (1.9) we arrive at

$$G(p) = L(p) = p, \quad p \in [0, 1].$$

This is only possible if either

$$b = 0, \quad M(p) = p \quad \text{in (2.19)}$$

or if

$$b = 1, \quad M_1(p) = M_2(p) = p \quad \text{in (2.20)}.$$

In both cases we get the solution (2.3). Now we assume that

$$\sum_{i=1}^n (G(p_i) - p_i) \neq 0 \quad \text{for some } P \in \Gamma_n \quad \text{or}$$

$$\sum_{j=1}^m (L(q_j) - q_j) \neq 0 \quad \text{for some } Q \in \Gamma_m.$$

Then (2.12) and (2.13) imply that $I(0) = 0$ so that in all cases where

$$G(p) \neq p \quad \text{or} \quad L(p) \neq p$$

we get that I is equal to I' and thus I is not dependent upon n and m (see (2.25)). Using $G(1) = L(1) = 1$ and the hypothesis that I is not constant we obtain from (2.19) to (2.24) the remaining solutions (2.1), (2.2), (2.4) and (2.5). Thus the Theorem is proven.

It is clear that we can obtain from Theorem 2 some new characterization theorems for information measures. For instance, we remark that the functions G and L given by (2.4) or (2.5) cannot be continuous simultaneously. Thus we get the following extension of results in Kannappan [3,4], Sharma and Taneja [2,6].

COROLLARY 3. If in addition to the hypotheses of Theorem 2, G and L are continuous then the only solutions (I, G, L) of (1.9) and (1.3) are given by (2.1), (2.2) and (2.3).

Corollary 3 implies immediately the following characterization theorem :

Let I_k be an (n, m) -weighted additive information measure where I_k, G_k, L_k have the sum property with continuous generating functions $I, G, L : [0, 1] \rightarrow \mathbb{R}$. If

$$I(0) = G(0) = L(0) = 0, \quad G(1) = L(1) = 1 \quad \text{and} \quad I\left(\frac{1}{2}\right) = \frac{1}{2}$$

then $I_k(p) = H_k^{(A, B)}(p)$ or $I_k(p) = H_k^C(p)$, $p \in \Gamma_k$. Here A, B, C are real constants with $A \neq B$.

Finally we give two interpretations of our result. If we put $b = 0$ into (2.1), (2.2) and (2.3) then we get - with unchanged $I(p)$ -

$$G(p) = p^A, \quad L(p) = p^B = p^B - p^A + p^A = \frac{1}{a} I(p) + p^A, \quad A \neq B,$$

$$G(p) = p^A, \quad L(p) = p^A, \quad A \neq 1,$$

$$G(p) = p, \quad L(p) = p,$$

respectively. Thus we may consider Corollary 3 as a justification for the fact that in the literature only two special forms of G and L were considered, namely (1.4) and (1.5).

On the other hand, the condition $b = 0$ in (2.1) and (2.2) implies that in Corollary 3 we may assume without loss of generality that G and L are continuous, non zero multiplicative functions. This result is analogous to a result concerning recursive measures of multiplicative type (Aczél and Ng, [11]).

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