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TIME - PERIODIC WEAK SOLUTIONS

ELIANA HENRIQUES DE BRITO

Instituto de Matemática Universidade Federal do Rio de Janeiro C.P. 68530, Rio de Janeiro, Brasil

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ABSTRACT. In continuing from previous papers, where we studied the existence and uniqueness of the global solution and its asymptotic behavior as time t goes to infinity, we now search for a time-periodic weak solution u(t) for the equation whose weak formulation in a Hilbert space H is

$$\frac{d}{dt} (u',v) + \delta(u',v) + \alpha b(u,v) + \beta a(u,v) + (G(u),v) = (h,v)$$

where: ' = d/dt;(,) is the inner product in H; b(u,v), a(u,v) are given forms on subspaces U \subset W, respectively, of H; $\delta > 0$, $\alpha > 0$, $\beta > 0$ are constants and $\alpha + \beta > 0$; G is the Gateaux derivative of a convex functional J: V \subset H + [0, ∞) for V = U, when $\alpha > 0$ and V = W when $\alpha = 0$, hence $\beta > 0$; v is a test function in V; h is a given function of t with values in H.

Application is given to nonlinear initial-boundary value problems in a bounded domain of R^n .

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1. INTRODUCTION.

In continuation of Brito [1], [2], where we studied existence and uniqueness of the global solution and its asymptotic behavior as time t goes to infinity, we now search for a time-periodic weak solution u(t), i.e., such that u(0) = u(T); u'(0) = u'(T)

for the equation whose weak formulation in a Hilbert space H is

$$\frac{d}{dt}(u',v) + \delta(u',v) + \alpha b(u,v) + \beta a(u,v) + (G(u),v) = (h,v) \quad (1.1)$$

where

= d/dt; (,) is the inner product in H; b(u,v), a(u,v)

are given forms on subspaces $U \subset W$, respectively, of H; $\delta > 0$, $\alpha > 0$, $\beta > 0$ are constants and $\alpha + \beta > 0$; G is the Gateaux derivative of a convex functional J: $V \subset H \rightarrow [0,\infty)$, for V = U, when $\alpha > 0$, and V = W, when $\alpha = 0$, hence $\beta > 0$; v is a

test function in V; h is a given function of t with values in H.

Application is given to initial-boundary value problems in a bounded domain Ω of \mathbb{R}^n for the following equations, in which p > 2 depends on n, $\alpha > 0$, $\beta > 0$, k > 0:

$$u'' + \delta u' - \Delta u + |u|^{p-2} u = h$$
 (1.2)

u" + δu' +
$$\alpha \Delta^2 u$$
 - β Δu + u + $|u|^{p-2}u$ = h (1.3)

$$u'' + \delta u' + \alpha \Delta^2 u - \{\beta + k \int_{\Omega} (\nabla u)^2 d\Omega\} \Delta u = h \qquad (1.4)$$

and the generalization of (1.4) in a Hilbert space H

$$u'' + \delta u' + \alpha A^{2} u + \beta A u + M(|A^{1/2}|^{2})A u = h$$
 (1.5)

for A a linear operator in H, M a real function.

For related problems, we refer to Biroli [3], Lovicar [4]; see also references in Brito [1].

2. PRELIMINARIES.

We consider three Hilbert spaces $U \subset W \subset H$ each continuously embedded and dense in the following.

We assume the injection $W \subset H$ compact.

Let (,) denote the inner product in H and | its norm.

Let a(u,v) and b(u,v) be two continuous, symmetric, bilinear forms in W and U, respectively. We shall write a(v) for a(v,v), b(v) for b(v,v). We shall assume that $(a(v))^{1/2}$ defines in W a norm equivalent to the norm of W and, similarly, that $(b(v))^{1/2}$ defines in U a norm equivalent to the norm of U.

Let c > 0 be such that

$$c|v|^2 \leq a(v)$$
 for v in W. (2.1)

Let A be a linear operator in H, with domain D(A) such that $U \subset D(A) \subset W$ and a(u,v) = (Au,v) for u in U, v in W b(u,v) = (Au, Av) for u,v in U $\delta > 0, \alpha > 0, \beta > 0$ are constants and $\alpha + \beta > 0$.

Assume

 $V = U \text{ if } \alpha > 0, \beta > 0;$ $V = W \text{ if } \alpha = 0, \beta > 0.$ Consider a convex functional

J: $V \neq [0,\infty)$ such that J(0) = 0.

Let G: V + H be the Gateaux derivative of J. We assume G is Gateaux differentiable, locally Lipschitz and G(0) = 0.

With these hypothesis, we have, from [2] Theorem 3.1 and [1] Lemma 2.1, respectively

THEOREM 2.1. Given u_0 in V, u_1 in H, h in $L^2(0,T;H)$, there is a unique function u such that

a)
$$u \in L^{\infty}(0,T;V); u' \in L^{\infty}(0,T;H); G(u) \in L^{\infty}(0,T;H)$$

b) for all v in V, u satisfies,
$$\frac{d}{dt}(u',v) + \delta(u',v) + \alpha b(u,v) + \beta a(u,v) + (G(u),v) = (h,v) \quad (2.2)$$

c) u satisfies the initial conditions

$$u(0) = u_0; u'(0) = u_1$$
 (2.3)

d) u satisfies the energy equation

$$E(t) + \delta \int_{0}^{t} |u'(s)|^{2} ds = E(0) + \int_{0}^{t} (h(s), u'(s)) ds \qquad (2.4)$$

where

$$2E(t) = |u'(t)|^2 + \alpha b(u(t)) + \beta a(u(t)) + 2J(u(t)).$$

THEOREM 2.2. In the conditions of Theorem 2.1, the map S: V×H
$$\rightarrow$$
 V×H given by $S(u_0,u_1) = (u(\iota),u'(t))$

is, for fixed t, (sequentially) weakly continuous (i.e., if $\phi_n > \phi$ weakly in V × H, then $s(\phi_n) > s(\phi)$ weakly in V × H).

We shall, further, assume that

$$2J(v) - (G(v),v) \le 0$$
 for v in V. (2.5)

3. EXISTENCE OF TIME-PERIODIC WEAK SOLUTIONS.

We shall refer to u(t) in the conditions of Theorem 2.1 as the solution of (2.2) with initial conditions (u_0,u_1) in $V \times H$, given by (2.3).

THEOREM 3.1. If $h \in C([0,T];H)$ there is at least one solution of (2.2) with initial condition in V × H such that

$$u(0) = u(T); u'(0) = u'(T).$$
 (3.1)

PROOF. Take v = u(t) in (2.2) multiplied by constant $2\gamma > 0$ and add it to the energy equation (2.4) differentiated and multiplied by 2, to obtain, with (2.5),

$$\frac{d}{dt} - \{ |u'|^2 + \alpha b(u) + \beta a(u) + 2J(u) + 2\gamma(u,u') \} + 2\gamma\{ |u'|^2 + \alpha b(u) + \beta a(u) + 2J(u) + 2\gamma(u,u') \} + 2(\delta - 2\gamma) [|u'|^2 + \gamma(u',u)] \le 2(h,u' + \gamma u).$$

For $0 < \gamma < \delta/2$, let

$$w(t) = |u' + \gamma u|^2 + \alpha b(u) + \beta a(u) + 2J(u). \qquad (3.2)$$

Then we have

w'(t) + $2\gamma w(t) \leq 2(h,u' + \gamma u) - 2(\delta - 2\gamma)(u',u' + \gamma u) +$

+
$$\gamma^2 \frac{d}{dt} |u|^2 + 2\gamma^3 |u|^2$$
.

The right-hand side of the above inequality is equal to

= $2(h,u' + \gamma u) - 2(\delta - 2\gamma)(u' + \gamma u,u' + \gamma u) + 2(\delta - \gamma)(\gamma u,u' + \gamma u).$

Therefore

$$w'(t) + 2\gamma w(t) \le 2(h, u' + \gamma u) + \frac{(\delta - \gamma)^2 \gamma^2 |u|^2}{2(\delta - 2\gamma)}$$
 (3.3)

Observing (2.1) and (3.2), we obtain

$$w(t) > |u' + \gamma u|^2 + \varepsilon |u|^2, \text{ with } \varepsilon = ac^2 + \beta c > 0.$$
(3.4)

By assumption, $\beta + \alpha > 0$.

We choose $0 < \gamma < \delta/2$ so that

$$\rho = \frac{\left(\delta - \gamma\right)^2 \gamma^2}{2 \epsilon \left(\delta - 2\gamma\right)} < 2\gamma.$$
(3.5)

This is possible, because it amounts to choosing $\boldsymbol{\gamma}$ so that

 $B(\gamma) = (\delta - \gamma)^2 \gamma - 4 \varepsilon (\delta - 2\gamma) < 0$

and $\lim_{\gamma \to 0} B(\gamma) = -4 \varepsilon \delta < 0.$ $\gamma \to 0$ It follows from (3.3), with (3.4), (3.5), that $w'(t) + 2\gamma w(t) \le 2|h(t)| \sqrt{w(t)} + \rho w(t).$

Hence for $0 \le t \le T$,

$$w(t) \leq F(t) \tag{3.6}$$

where

$$F(t) = e^{-2\gamma t} \{w(0) + \int_{0}^{t} e^{2\gamma s} [2|h|/w + \rho w] ds\}.$$
(3.7)

Therefore, because of (3.6), we have $F'(t) \leq (\rho - 2\gamma)F(t) + 2|h(t)|/F(t).$

Let F(t) = r, with

r > max
$$\frac{2|h(t)|}{2\gamma - \rho}$$
.
0 < t < T

Then $F'(t) \leq 0$. It follows, using (3.6), (3.7), that if $w(0) \leq r$ then $w(t) \leq r$, for $0 \leq t \leq T$.

Consider

 $K = \{(u_0, u_1) \in V \times H; |u_1 + \gamma u_0|^2 + \alpha b(u_0) + \beta a(u_0) + 2J(u_0) \leq r\}.$ We proved that the map S: V × H + V × H given by

$$S(u_0, u_1) = (u(T), u'(T))$$

takes K into K.

It is easy to check that K is a nonempty, closed, bounded, convex subset of V \times H.

The fact that S has a fixed point, i.e., that (3.1) holds for some $(u_0,u_1) \in K$, now follows from Theorem 2.2 as a consequence of the well-known fixed point Theorem:

Let B be a separable, reflexive Banach space, K a nonempty closed, bounded convex subset of B, and S a (sequentially) weakly continuous operator of K into K. Then S has at least one fixed point in K.

4. APPLICATIONS.

We devote this Section to applications of Theorem 3.1 involving initial-boundary value problems in a bounded domain Ω with regular boundary in Rⁿ for equations (1.2), (1.3), (1.4).

In what follows $H = L^{2}(\Omega), W = H_{\Omega}^{1}(\Omega), \text{ and}$

$$\mathbf{a}(\mathbf{u},\mathbf{v}) = (\nabla \mathbf{u},\nabla \mathbf{v}) = \int_{\Omega} \nabla \mathbf{u} \nabla \mathbf{v} d\Omega.$$

Let

$$A = -\Delta, \quad D(A) = H_0^1(\Omega) \cap H^2(\Omega),$$
$$b(u,v) = (\Delta u, \Delta v), \quad U = H_0^1(\Omega) \cap H^2(\Omega).$$

Note that similar results are obtained if we suppose $U = H_0^2(\Omega)$.

For $\delta > 0$, h ϵ C([0,T]; H), we have EXAMPLE 4.1. Let $\alpha = 0$, $\beta = 1$, $V = W = H_0^1(\Omega)$ and

$$J(u) = \frac{1}{p} \left| u \right|^{p} L^{p}(\Omega) \text{ for } u \text{ in } V$$

where 2 if <math>n > 2; p > 2 if n < 2. Then J(u) is well-defined in V and

$$G(u) = |u|^{p-2} u \in H.$$

We refer to [2], example 5.1, for the proof. It is clear that (2.5) holds. Therefore Theorem 3.1 ensures existence of a T-periodic weak solution of

 $u'' + \delta u' - \Delta u + |u|^{p-2}u = h.$ EXAMPLE 4.2. For $\alpha > 0$, $\beta > 0$, $V = U = H_0^1(\Omega) \cap H^2(\Omega)$ (or $H_0^2(\Omega)$), let

$$J(u) = \frac{1}{p} \left| u \right|_{L^{p}(\Omega)}^{p} + \frac{1}{2} \left| u \right|_{L^{2}(\Omega)}^{2} \text{ for } u \text{ in } V$$

where

$$2 if $n > 4$; $p > 2$ if $n < 4$.$$

Then J(u) is well-defined in V and

 $G(u) = |u|^{p-2}u + u \in H.$

We refer to [2], example 5.2, for the proof. It is clear that (2.5) holds. Then Theorem 3.1 ensures existence of a T-periodic weak solution of

$$u'' + \delta u' + \alpha \Delta^2 u - \beta \Delta u + u + |u|^{p-2} u = h.$$

let

$$J(u) = \frac{k}{4} (a(u))^2 \text{ for } u \text{ in } V_{4}$$

Then

 $C(u) = -ka(u)\Delta u \in H.$

We refer to [2], Example 5.3, for the proof. It is clear that (2.5) holds.

Thus Theorem 3.1. ensures existence of a T-periodic weak solution of

$$u'' + \delta u' + \alpha \Delta^2 u - \{\beta + k \int_{\Omega} (\nabla u)^2 d\Omega\} \Delta u = h.$$

Generalizing, let M be a C^1 function such that, for s > 0,

$$M(s) > k$$
 and $M'(s) > 0$

Take V = U and A as in Section 1. Let

$$J(u) = \frac{1}{2} \int_{0}^{a(u)} M(s) ds \text{ for } u \text{ in } V.$$

Then

 $G(u) = M(a(u)), Au \in H.$

We refer to [2], Example 5.3, for the proof. It is clear that (2.5) holds. Therefore Theorem 3.1 ensures existence of a T-periodic weak solution of **u**"

$$' + \delta u' + \alpha A^2 u + [\beta + M(|A^{1/2}u|^2)] Au = h.$$

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