ON α -CONVEX FUNCTIONS OF ORDER β WITH m -FOLD SYMMETRY

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ABSTRACT. This note is a continuation of the previous work [1,2,3]. First we get a new subordination for α -convex functions of order β when $\alpha=1-2\beta$, which implies the rotation theorem for $(1-2\beta)/m$ -convex functions of order β with m -fold symmetry. Then we extend the known results on α -convex functions of order β to the functions with m -fold symmetry. In particular, we give the sharp order of convexity of α -convex functions of order β with m -fold symmetry for $\alpha > 1$, which is analogous in sharpness to a result given by Miller, Mocanu and Reade [1].

KEYWORDS AND PHRASES. Subordination, α -convex functions of order β , m -fold symmetry, rotation theorem, order of convexity, distortion theorems.

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1. INTRODUCTION.

Let $J_m(\alpha, \beta)$ be the class of α -convex functions of order β with m -fold symmetry, where $\alpha > 0$, $0 \leq \beta < 1$ and $m=1, 2, \dots$. That is, it consists of analytic functions $f(z) = z + \sum_{n=1}^{\infty} a_{nm+1} z^{nm+1}$ in the unit disk $D = \{z : |z| < 1\}$ with $f(z)f'(z)/z \neq 0$ and

$$\operatorname{Re}\{(1-\alpha)zf'(z)/f(z) + \alpha(1+zf''(z)/f'(z))\} > \beta. \quad (1.1)$$

In [1], Miller, Mocanu and Reade studied the class $J(\alpha, 0) = J_1(\alpha, 0)$. Liu [2] and we [3] discussed the class $J(\alpha, \beta) = J_1(\alpha, \beta)$. Liu got the sharp bounds of $|f(z)|$, $|a_3 - \mu a_2^2|$ ($-\infty < \mu < +\infty$) and $|\arg f'(z)|$ for $\alpha=0, 1$. In [3], we obtained a subordination result for $J(\alpha, \beta)$, some distortion theorems, etc.

This note is a continuation of previous work. First we get a new subordination theorem for the class $J(1-2\beta, \beta)$, which implies the rotation theorem for

$J_m((1-2\beta)/m, \beta)$. Then we extend known results on $J(\alpha, \beta)$ to the class $J_m(\alpha, \beta)$. In particular, we give the sharp order of convexity of functions in the class

$J_m(\alpha, \beta)$ for $\alpha > 1$, which is analogous in sharpness to a result given by Miller, Mocanu and Reade [1].

2. SUBORDINATION AND DISTORTION PROPERTIES.

At first, we establish a homeomorphic relation between $J_m(\alpha, \beta)$ and $J(m\alpha, \beta)$.

LEMMA 1. $f(z) \in J_m(\alpha, \beta)$ if and only if $g(z) \in J(m\alpha, \beta)$, where $g(z) = f(z^{1/m})^m$.

PROOF. If $f(z) \in J_m(\alpha, \beta)$, then $g(z)$ is also analytic in D . It is not difficult to show that $g(z)g'(z)/z \neq 0$ and

$$\begin{aligned} & (1-\alpha)zf'(z)/f(z) + \alpha(1+zf''(z)/f'(z)) \\ &= (1-m\alpha)ug'(u)/g(u) + m\alpha(1+ug''(u)/g'(u)) \end{aligned}$$

in D , where $u = z^m$. Hence $g(z) \in J(m\alpha, \beta)$. Similarly we can prove

$f(z) = g(z^m)^{1/m} \in J_m(\alpha, \beta)$ if $g(z) \in J(m\alpha, \beta)$. This completes the proof.

It is well known that $G(z) \in J(0, \beta)$ if and only if there is a probability measure $\mu(x)$ on the unit circle $X = \{x: |x|=1\}$ such that

$$G(z) = z \exp\{2(1-\beta) \int_X -\log(1-xz) d\mu(x)\}.$$

This implies, by Lemma 1, that $F(z) \in J_m(0, \beta)$ if and only if there is a probability measure $\mu(x)$ on X such that

$$F(z) = z \exp\{2(1-\beta)m^{-1} \int_X -\log(1-xz^m) d\mu(x)\}. \quad (2.2)$$

Because $g(z) \in J(m\alpha, \beta)$ if and only if there is a $G(z) \in J(0, \beta)$ such that [2]

$$g(z) = \{\alpha^{-1}m^{-1} \int_0^z u^{-1} G(u)^{1/m} du\}^{m\alpha},$$

we have for $\alpha > 0$ that $f(z) \in J_m(\alpha, \beta)$ if and only if there is a $F(z) \in J_m(0, \beta)$ such that

$$f(z) = \{\alpha^{-1} \int_0^z u^{-1} F(u)^{1/\alpha} du\}^\alpha. \quad (2.3)$$

From (2.2) and (2.3), we obtain the following result.

If $f(z) \in J_m(\alpha, \beta)$ and $|z|=r < 1$, then

$$e^{-i\pi/m} k_m(\alpha, \beta, re^{i\pi/m}) < |f(z)| < k_m(\alpha, \beta, r), \quad (2.4)$$

where

$$k_m(\alpha, \beta, z) = \begin{cases} z(1-z^m)^{-2(1-\beta)/m} & (\alpha=0) \\ \alpha^{-1} \int_0^z u^{-1+1/\alpha} (1-u^m)^{-2(1-\beta)/m\alpha} du & (\alpha>0) \end{cases} \quad (2.5)$$

is the α -convex Koebe function of order β with m -fold symmetry.

Specifically we denote $k_1(\alpha, \beta, z)$ by $k(\alpha, \beta, z)$.

In order to state our subordination theorem, we shall make use of the following lemma.

LEMMA 2. Let $\log q(z)$ be a convex univalent function in D and $p_i(z) \prec q(z)$ ($i=1,2,\dots,n$). Then for $\lambda_i > 0$ and $\sum_{i=1}^n \lambda_i = 1$

$$\prod_{i=1}^n p_i(z)^{\lambda_i} \prec q(z).$$

PROOF. Since $\log q(z)$ is a convex function and $p_i(z) \prec q(z)$, we have $p_i(z) \neq 0$ and $\log p_i(z) \prec \log q(z)$, which implies

$$\log p_i(D) \subset \log q(D).$$

From the fact that $\log q(D)$ is a convex domain, we get for each $z \in D$

$$\sum_{i=1}^n \lambda_i \log p_i(z) \in \log q(D),$$

and then

$$\sum_{i=1}^n \lambda_i \log p_i(z) \prec \log q(z),$$

which is equivalent to the desired result.

COROLLARY 1. If $p_i(z) \prec (1-bz)/(1-az)$ ($i=1,2,\dots,n$, $-1 \leq a, b \leq 1$), then for $\lambda_i > 0$ and $\sum_{i=1}^n \lambda_i = 1$ we have

$$\prod_{i=1}^n p_i(z)^{\lambda_i} \prec (1-bz)/(1-az).$$

PROOF. For $a=b$, the result is trivial. For $a \neq b$, we know $\log(1-bz) - \log(1-az)$ is a convex function. Hence the required result follows from Lemma 2.

This corollary and some of its applications may be found elsewhere [4].

THEOREM 1. Let $g(z) \in J(1-2\beta, \beta)$ and $0 < \lambda < 1$, then

$$g'(z)^\lambda (g(z)/z)^{1-2\lambda} \prec 1/(1-z). \quad (2.6)$$

In particular, we have

$$g'(z) \prec 1/(1-z)^2, \quad (2.7)$$

$$g(z)/z \prec 1/(1-z). \quad (2.8)$$

PROOF. First we prove (2.8).

If $\beta = \frac{1}{2}$, then (1.1) becomes $\operatorname{Re}\{zg'(z)/g(z)\} > \frac{1}{2}$, which gives

$$zg'(z)/g(z) \prec 1/(1-z).$$

If $\beta < \frac{1}{2}$, we know [3]

$$zg'(z)/g(z) \prec zk'(1-2\beta, \beta, z)/k(1-2\beta, \beta, z) = 1/(1-z).$$

In both of these cases, we have

$$zg'(z)/g(z) - 1 \prec z/(1-z).$$

Since $z/(1-z)$ is convex [5],

$$\int_0^z u^{-1} (ug'(u)/g(u) - 1) du \prec \int_0^z 1/(1-u) du.$$

That is, $\log g(z) \prec \log 1/(1-z)$, which is equivalent to (2.8).

By using Corollary 1 for $p_1(z) = zg'(z)/g(z)$, $p_2(z) = g(z)/z$, $\lambda_1 = \lambda$, and $\lambda_2 = 1 - \lambda$, we obtain (2.6). The proof is completed.

THEOREM 2. Let $f(z) \in J_m((1-2\beta)/m, \beta)$, $0 < \lambda < 1$ and $|z| = r < 1$, then we have the sharp estimates

$$1/(1-r^m) |f(z)|^\lambda |f(z)/z|^{m(1-\lambda)-\lambda} \leq 1/(1-r^m), \quad (2.9)$$

$$|\lambda \arg f'(z) + (m(1-\lambda) - \lambda) \arg(f(z)/z)| \leq \arcsin r^m \quad (2.10)$$

PROOF. Let $g(z) = f(z)^{1/m}$, we know $g(z) \in J(1-2\beta, \beta)$ from Lemma 1 and $zf'(z)/f(z) = ug'(u)/g(u)$, where $u = z^m$. Let

$$p(z) = g'(z)^\lambda (g(z)/z)^{1-2\lambda}, p_1(z) = f'(z)^\lambda (f(z)/z)^{(1-\lambda)m-\lambda},$$

then

$$p_1(z) = (zf'(z)/f(z))^\lambda (f(z)/z)^{(1-\lambda)m} = (ug'(u)/g(u))^\lambda (g(u)/u)^{1-\lambda} = p(u).$$

From Theorem 1 and the principle of subordination, we have

$p(|u| < R) \subset q(|u| < R)$ for every R ($0 < R < 1$), where $q(z) = 1/(1-z)$. This implies $p_1(|z| < r) \subset q_1(|z| < r)$ for every r ($0 < r < 1$), where $q_1(z) = 1/(1-z^m)$, which gives the results. This completes the proof of theorem 2.

The inequality (2.10) contains the following rotation theorem for $J_m((1-2\beta)/m, \beta)$.

COROLLARY 2. If $f(z) \in J_m((1-2\beta)/m, \beta)$ and $|z| = r$, then

$$|\arg f'(z)| \leq (m+1) \arcsin r^m / m. \quad (2.11)$$

The following subordination is due to Liu [2].

$$g'(z)^\alpha (g(z)/z)^{1-\alpha} \prec (1-z)^{-2(1-\beta)} \quad (2.12)$$

whenever $g(z) \in J(\alpha, \beta)$. In [3] we found that if $g(z) \in J(\alpha, \beta)$, then

$$zg'(z)/g(z) \prec zk'(\alpha, \beta, z)/k(\alpha, \beta, z). \quad (2.13)$$

By using a method similar to that used in the proof of theorem 2, we can obtain the following theorems from (2.12) and (2.13). Here we omit most of their proofs. When $m=1$, most of the following results were given in [2] and [3] respectively.

THEOREM 3. Let $f(z) \in J_m(\alpha, \beta)$, $|z|=r < 1$, then we have sharp results

$$r^{1-\alpha/(1+r^m)^{2(1-\beta)/m}} |f'(z)|^\alpha |f(z)|^{1-\alpha} \leq r^{1-\alpha/(1-r^m)^{2(1-\beta)/m}}, \quad (2.14)$$

$$|\arg\{f'(z)^\alpha (f(z)/z)^{1-\alpha}\}| \leq 2(1-\beta)m^{-1} \arcsin r^m, \quad (2.15)$$

$$\operatorname{Re}\{f'(z)^\alpha (f(z)/z)^{1-\alpha}\} \geq 2^{-2(1-\beta)/m}. \quad (2.16)$$

THEOREM 4. Let $f(z) \in J_m(\alpha, \beta)$, $|z|=r < 1$, then we have the sharp inequalities

$$\begin{aligned} r e^{i\pi/m} k'_m(\alpha, \beta, r e^{i\pi/m}) / k_m(\alpha, \beta, r e^{i\pi/m}) &\leq |zf'(z)/f(z)| \\ &\leq rk'_m(\alpha, \beta, r) / k_m(\alpha, \beta, r), \end{aligned} \quad (2.17)$$

$$\begin{aligned} |\arg\{zf'(z)/f(z)\}| &\leq \max_{|z|=r} \arg\{zk'_m(\alpha, \beta, z)/k_m(\alpha, \beta, z)\}. \\ |z| &= r \end{aligned} \quad (2.18)$$

PROOF. We give an outline of the proof of (2.17). Let

$$p(z) = zf'(z)/f(z), \quad q(z) = zk'_m(\alpha, \beta, z)/k_m(\alpha, \beta, z).$$

We know that $q(z^{1/m})$ is univalent in D [3]. As the proof of theorem 2, we can get

$$p(|z| \leq r) \subset q(|z| \leq r) \quad (0 \leq r < 1).$$

Thus for $|z|=r$ we obtain

$$\min_{|z|=r} |q(z)| \leq |zf'(z)/f(z)| \leq \max_{|z|=r} |q(z)|.$$

We prove $\max_{|z|=r} |q(z)| = q(r)$ and $\min_{|z|=r} |q(z)| = q(re^{i\pi/m})$.

Let $q(z) = 1 + B_1 z^m + B_2 z^{2m} + \dots$, it follows from

$$q(z) + \alpha z q'(z) / q(z) = (1 + (1-2\beta)z^m) / (1-z^m)$$

that

$$(1+m\alpha)B_n = 2(1-\beta) + \sum_{k=1}^{n-1} (2-2\beta-B_{n-k})\beta_k.$$

By using the fact that $\operatorname{Re} q(z) > \beta$ [3], we have $|B_k| \leq 2(1-\beta)$ [6]. Hence we get

$B_n > 0$ ($n=1, 2, \dots$) by induction and also $\max_{|z|=r} |q(z)| = q(r)$.

Because the coefficients of $q(z)$ are all real and $q(z)$ is m -fold symmetric, we can obtain $\min_{|z|=r} |q(z)| = q(re^{i\pi/m})$ by proving

$$|q(re^{i\theta})| > q(re^{i\pi/m}) \quad (0 < \theta < \pi/m). \quad (2.19)$$

If $\alpha=0$, it is obvious that (2.19) is true for $q(z) = (1 + (1-2\beta)z^m)/(1-z^m)$.

If $\alpha > 0$, we have

$$\begin{aligned} q(z) &= (z(1-z^m)^{-2(1-\beta)/m})^{1/\alpha} (\alpha^{-1} \int_0^z u^{1/\alpha-1} (1-u^m)^{-2(1-\beta)/m\alpha} du)^{-1}, \\ |q(re^{i\theta})| &= (\alpha^{-1} \int_0^r u^{1/\alpha-1} (1-u^m)^{-2(1-\beta)/m\alpha} du)^{-1} \\ &< \int_0^r t^{1/\alpha-1} (1-2t^m \cos m\theta + t^{2m})^{-(1-\beta)/m\alpha} dt, \end{aligned}$$

which implies that

$$\begin{aligned} |q(re^{i\theta})| &> \frac{\alpha(r(1-2r^m \cos m\theta + r^{2m})^{-(1-\beta)/m})^{1/\alpha}}{\int_0^r t^{1/\alpha-1} (1-2t^m \cos m\theta + t^{2m})^{-(1-\beta)/m\alpha} dt} \\ q(re^{i\pi/m}) &= \alpha(r(1+r^m)^{-2(1-\beta)/m})^{1/\alpha} / \int_0^r t^{1/\alpha-1} (1+t^m)^{-2(1-\beta)/m\alpha} dt. \end{aligned}$$

Let $I(\theta) =$

$$(1-2r^m \cos m\theta + r^{2m})^{-(1-\beta)/m\alpha} \int_0^r t^{1/\alpha-1} (1-2t^m \cos m\theta + t^{2m})^{-(1-\beta)/m\alpha} dt.$$

We can verify $I'(\theta) > 0$ ($0 < \theta < \pi/m$), which implies the desired result. The proof of (2.17) is now complete.

From (2.4) and (2.17), we get the following distortion result.

COROLLARY 3. If $f(z) \in J_m(\alpha, \beta)$, $|z| = r < 1$, then

$$k'_m(\alpha, \beta, re^{i\pi/m}) < |f'(z)| < k'_m(\alpha, \beta, r).$$

From (2.13), we can also obtain the sharp order of starlikeness for functions in $J_m(\alpha, \beta)$.

THEOREM 5. Let $f(z) \in J_m(\alpha, \beta)$. Then $f(z) \in J_m(0, s_m(\alpha, \beta))$, that is, $f(z)$ is starlike of order $s_m(\alpha, \beta)$, where

$$s_m(\alpha, \beta) = \min_{0 < \theta < 2\pi/m} \operatorname{Re}\{e^{i\theta} k'_m(\alpha, \beta, e^{i\theta}) / k'_m(\alpha, \beta, e^{i\theta})\} > \beta.$$

Miller, Mocanu and Reade [1] proved that $f(z)$ is a convex function if $f(z) \in J(\alpha, 0)$ and $\alpha > 1$. By making use of theorem 5, we get the following sharp order of convexity, which is analogous in sharpness to a result in [1].

COROLLARY 4. If $f(z) \in J_m(\alpha, \beta)$ and $\alpha > 1$, then

$f(z) \in J_m(1, \beta/\alpha + (1-1/\alpha)s_m(\alpha, \beta))$, that is, $f(z)$ is convex of order

$\beta/\alpha + (1-1/\alpha)s_m(\alpha, \beta) \quad (\geq \beta)$.

By using the method we used in [3], we can easily get the following covering theorem from (2.4).

THEOREM 6. Let $w = f(z) \in J_m(\alpha, \beta)$. Then we have the sharp result $f(D) \supset \{w: |w| < d_m(\alpha, \beta)\}$, where

$$d_m(\alpha, \beta) = \begin{cases} 2^{-2(1-\beta)/m} & (\alpha=0) \\ F(1/m\alpha, 2(1-\beta)/m\alpha, 1+1/m\alpha; -1) & (\alpha>0) \end{cases}$$

and F is the hypergeometric function.

Finally, we note a coefficient inequality, which can be deduced from (2.1) and a similar result on $J(\alpha, \beta)$ given in [2].

THEOREM 7. Let $f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \dots \in J_m(\alpha, \beta)$, then we have the sharp inequalities

$$\begin{cases} |a_{2m+1} - \lambda a_{m+1}^2| < \begin{cases} \frac{(1-\beta)^2}{m(1+2m\alpha)} \left\{ \frac{2m+6m^2\alpha+m^3\alpha^2-(2\lambda-1)(1+2m\alpha)}{m(1+m\alpha)^2} + \frac{\beta}{1-\beta} \right\} & -\infty < \lambda < a; \\ (1-\beta)/(m(1+2m\alpha)) & a < \lambda < b; \\ \frac{(1-\beta)^2}{m(1+2m\alpha)} \{ (m-1+2\lambda)(1+2m\alpha)/(m(1+m\alpha)^3) - \beta/(1-\beta) \} & b < \lambda < +\infty, \end{cases} \end{cases}$$

where

$$a = \frac{1}{2} + \frac{1}{2}m^2\alpha/(1+2m\alpha), \quad b = \frac{1}{2} + \frac{1}{2}m^2\alpha/(1+2m\alpha) + \frac{1}{2}m(1+m\alpha)^2/((1+2m\alpha)(1-\beta)).$$

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