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STRONG OSCILLATIONS FOR SECOND ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we establish some strongly oscillation theorems for nonlinear second order functional differential equation

$$x''(t) + p(t)\, f(x(t),\, x(g(t))) = 0$$

without assuming that $g(t)$ is retarded or advanced.

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1. INTRODUCTION. We consider the second order nonlinear functional differential equation

$$x''(t) + p(t)\, f(x(t)\ ,\ x(g(t))) = 0 \tag{1.1}$$

where $p(t),\ g(t) \in C([t_0, \infty), R)$, $g(t) \to \infty$ as $t \to \infty$ and $f(u,v) \in C(R,R)$ and has the sign of u and v when they have the same sign. We shall restrict our attention to solutions of (1.1) which exist on some positive half-line. A nontrivial solution $x(t)$ is called oscillatory if $x(t)$ has an unbounded set of zeros, and otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if every solution of (1.1) is oscillatory.

Oscillation theory for (1.1) has been developed by many authors. Bradley [1], Chiou [2], Erbe [3] Gollwitzer [4], Ladas [5], Travis [6], Waltman [7], Wong [8] and references therein. It is wellknown theorem of Wintner [9] and Leighton [10] that the linear equation

$$x''(t) + p(t)\, x(t) = 0$$

is oscillatory if $\int^{\infty} p(t)dt = \infty$, even $p(t)$ is not assumed nonnegative. Bradley [1] and Waltman [7] demonstrated that the equation

$$x''(t) + p(t)\, x(g(t)) = 0 \tag{1.2}$$

is oscillatory if $p(t) \geq 0$ and $\int^{\infty} p(t)\, dt = \infty$. Travis [6] constructed a counterexample showing that the Leighton-Wintner oscillation theorem can not be extended to Equation (1.2) unless $p(t) \geq 0$. The author [11] extended Bradley-Waltman oscillation theorem to (1.1) i.e. if $p(t) \geq 0$ and $\int^{\infty} p(t)dt = \infty$, then (1.1) is ocillatory.

The purpose of this paper is to establish some strongly oscillation criteria for (1.1). We are primarily interested in the case when $p(t) \geq 0$, $\int^{\infty} p(t)dt < \infty$, are satisfied.

Considering the equation

$$x''(t) + \lambda p(t)\, x(t) = 0, \tag{1.3}$$

We shall call p(t) a strongly oscillatory coefficient if (1.3) is oscillatory for all positive λ. If $p(t) \geq 0$, Nehari [12] shows that

$$\limsup_{t \to \infty} t \int_t^{\infty} p(s)ds = \infty$$

is a necessary and sufficient condition for p(t) to be a strongly oscillatory coefficient. In general, motivated by Nehari, we define as follows: Equation (1.1) is said to be strongly oscillatory if the related equation of (1.1)

$$x''(t) + \lambda p(t)\, f(x(t), x(g(t))) = 0 \tag{1.4}$$

is osillatory for all positive λ .

2. MAIN RESULTS.

For Equation (1.1) the following conditions are assumed to hold throughout the paper:

(i) $p(t) \geq 0$ and there exists $h(t) \leq \min(g(t), t)$ such that $0 < k \leq h'(t)$ where k is a constant.
(ii) there exists $m > 0$ such that $|u| \geq m$ implies

$$\liminf_{|v| \to \infty} \frac{|f(u,v)|}{|\phi(v)|} \geq \varepsilon_1 > 0,$$

where $\phi(v) \in C'(R)$, $v\phi(v) > 0$ and $\phi'(v) \neq 0$ for $v \neq 0$, and $\lim_{|v| \to \infty} \phi'(v) \geq \delta > 0$ where ε_1 and δ are constants.

We begin with a Lemma which needed in establishing our results.

LEMMA 2.1. Suppose that for $\lambda = \lambda_o > 0$ Equation (1.1) has a nonoscillatory solution $x(t)$. Then the following inequality holds for all large t,

$$w(t) \geq \sigma \int_t^\infty w^2(s)\,ds + \lambda_o \varepsilon \int_t^\infty p(s)ds, \tag{2.1}$$

where $w(t) = x'(t)/\phi(x(h(t)))$, σ and ε are positive constants.

PROOF. Assume that Equation (1.4) at $\lambda = \lambda_o$, has a nonoscillatory solution $x(t) > 0$ for $t \geq t_o > 0$. A similar proof will hold if $x(t) < 0$ for $t \geq t_o$. It is easy to verify that $x''(t) < 0$ and $x'(t) > 0$ for all large t. Let $w(t) = x'(t)/\phi(x(h(t)))$, then

$$w'(t) = -\lambda_o p(t) \frac{f(x(t),\, x(g(t)))}{\phi(x(h(t)))} - \frac{\phi'(x(h(t)))x'(h(t))h'(t)}{\phi(x(h(f)))} w(t).$$

Since $x'(t) > 0$ for large t, $\lim_{t\to\infty} x(t)$ exists either as a finite or infinite limit. If $\lim_{t\to\infty} x(t) = \alpha$ is finite, then

$$\lim_{t\to\infty} \frac{f(x(t),\, x(g(t)))}{\varphi(x(g(t)))} = \frac{f(\alpha,\alpha)}{\phi(\alpha)} = \varepsilon_2 > 0.$$

If $\lim_{t\to\infty} x(t) = \infty$, then by (ii) we have that

$$\frac{f(x(t),\, x(g(t)))}{\phi(x(g(t)))} \geq \varepsilon_1$$

for large t. In either case, for sufficiently large t, we have that

$$\frac{f(x(t),\, x(g(t)))}{\phi(x(g(t)))} \geq \varepsilon\,, \quad \text{where} \quad \varepsilon = \min\,(\varepsilon_1,\, \varepsilon_2). \tag{2.2}$$

Since $x(t)$ is increasing, for large t we have that

$$p(t) \frac{f(x(t),\, x(g(t)))}{\phi(x(h(t)))} \geq p(t) \frac{f(x(t),\, x(g(t)))}{\phi(x(g(t)))} \geq \lambda_o \varepsilon p(t),$$

and in view of $x''(t) < 0$ for large t and (ii) we see that

$$\frac{\phi'(x(h(t)))x'(h(t))h'(t)}{\phi(x(h(t)))} w(t) \geq \frac{\phi'(x(h(t)))x'(t)h'(t)}{\phi(x(h(t)))} \geq k\delta w^2(t) = \sigma w^2(t)$$

Thus for $t \geq t_1 \geq t_o$,

$$w'(t) + \sigma\, w^2(t) + \lambda_o \varepsilon\, p(t) \leq 0 \tag{2.3}$$

and

$$w'(t) + \sigma\, w^2(t) \leq 0. \tag{2.4}$$

From (2.4), we have that

$$\frac{d}{dt}\left(-\frac{1}{w(t)} + \sigma t\right) \leq 0.$$

Integrating the above inequality from t_1 to t we obtain that

$$0 < w(t) \leq \frac{1}{\sigma t + b} \quad . \tag{2.5}$$

where $b = \frac{1}{w(t_1)} - kt_1$. From (2.5) we have that

$$\lim_{t \to \infty} w(t) = 0. \tag{2.6}$$

Integrating (2.3) from t_1 to t, then letting $t \to \infty$ we obtain that (2.1) holds for all large.

We introduce the function sequence $\{p_n(t)\}$, $n = 0,1,2,\ldots$, defined as follows:

$$p_o(t) = p(t), \quad p_1(t) = \int_t^\infty p_o(s)ds,$$

$$p_{n+1}(t) = \int_t^\infty p_n^2 (s)\, ds, \qquad n = 1,2,3,\ldots \; . \tag{2.7}$$

THEOREM 2.2. Assume that one of the following conditions holds,

(I_1) there is an integer $m \geq 1$ such that $p_n(t)$ is defined for $n = 1,2,\ldots,m$, and

$$\lim_{t \to \infty} \sup tp_m(t) = \infty; \tag{2.8}$$

(I_2) there is an integer $m \geq 2$ such that $p_n(t)$ is defined for $n = 1,2,\ldots, m-1$, but $p_m(t)$ does not exist, i.e.

$$\int^\infty p_{m-1}^2(t)\, dt = \infty.$$

Then equation (1.1) is strongly oscillatory.

PROOF. Assume to the contrary that Equation (1.4) at $\lambda=\lambda_o > 0$, has a non-oscillatory solution $x(t) > 0$ for $t \geq t_o \geq 0$. A similar argument holds when $x(t) < 0$ for $t \geq t_o \geq 0$. Let $w(t) = x'(t)/\phi(x(h(t)))$. As in the proof of Lemma 2.1, we can obtain

$$w'(t) + \sigma w^2(t) + \lambda_o \varepsilon p_o(t) \leq 0, \quad t \geq t_1 \geq t_o. \tag{2.9}$$

Suppose $m = 1$. Define $u_o(t) = \sigma w(t)$; then

$$u_o'(t) + u_o^2(t) + \xi_o p_o(t) \leq 0, \quad t \geq t_1 \geq t_o, \tag{2.10}$$

where $\xi_o = \lambda_o \varepsilon\sigma$. However, by a well-known theorem of Wintner [13] this implies the equation

$$y''(t) + \xi_o p_o(t)y(t) = 0 \tag{2.11}$$

is nonoscillatory. This contradicts the fact that the condition (I_1) implies $p_o(t)$ is a strongly oscillatory coefficient.

If $m > 1$, integrating (2.10) from t_1 to t we obtain

$$u_o(t) - u(t_1) + \int_{t_1}^{t} u_o^2(s)ds + \xi_o \int_{t_1}^{t} p_o(s)ds \leq 0 . \qquad (2.12)$$

Since $\lim_{t \to \infty} u_o(t) = \lim_{t \to \infty} \sigma w(t)=0$, , from (2.12) we get

$$u_o(t) \geq u_1(t) + \xi_o p_1(t), \quad t \geq t_1, \qquad (2.13)$$

where $u_1(t) = \int_t^\infty u_o^2(s)ds$. Then (2.13) implies that

$$u_1'(t) = - u_o^2(t) \leq - u_1^2(t) - 2\xi_o u_1(t)p_1(t) - \xi_o^2 p_1^2(t).$$

Hence

$$u_1'(t) + u_1^2(t) + \xi_1 p_1^2(t) \leq 0, \qquad (2.14)$$

where $\xi_1 = \xi_o^2$.

If $m = 2$, then by the Nehari theorem, the condition (I_1) implies equation

$$y''(t) + \xi_1 p_1^2 (t)y(t) = 0 \qquad (2.15)$$

is oscillatory, contradicting (2.14).

If the condition (I_2) is satisfied, then by the Leighton-Wintner theorem we have that equation (2.15) is oscillatory, again a contradiction.

When $m > 2$, we obtain inductively that

$$u_{m-1}'(t) + u_{m-1}^2(t) + \xi_{m-1} p_{m-1}^2 (t) \leq 0, \quad t \geq t^* \geq t_1 . \qquad (2.16)$$

where $u_{m-1}(t) = \int_t^\infty u_{m-2}^2(s)ds$, ξ_{m-1} is a constant and $p_{m-1}(t)$ is defined by (2.7). Applying the Wintner theorem again, it follows that equation

$$y''(t) + \xi_{m-1} p_{m-1}^2(t)y(t) = 0 \qquad (2.17)$$

is nonoscillatory. But this contradicts again the fact that the condition (I_1) or (I_2) implies that equation (2.17) is oscillatory. The proof is thus complete.

REMARK. Theorem 2.2 includes Theorem 2.2 in [6] as a special case, i.e. $\phi(u)=u$ and $m = 1$.

Consider the function sequence $\{q_n(t,\xi,\eta)\}$, $n=1,2,\ldots$, which is defined as follows:

$$q_o(t,\eta) = \eta \int_t^\infty p(s)ds, \quad q_1(t,\xi,\eta) = \xi \int_t^\infty q_o^2(s,\eta)ds + q_o(t,\eta),$$

$$q_n(t,\xi,\eta) = \xi \int_t^\infty q_{n-1}^2 (s,\xi,\eta)ds + q_o(t,\eta), \quad t \geq t_o, \ n = 2,3,\ldots . \qquad (2.18)$$

where ξ and η are positive constants.

THEOREM 2.3. For all positive constants ξ and η assume that any one of following conditions is satisfied:

(II_1) there is a positive integer m such that $q_n(t,\xi,\eta)$ is defined for $n=1,2,\ldots, m-1$, but $q_m(t,\xi,\eta)$ does not exist;

(II_2) $q_n(t,\xi,\eta)$ is defined for $n=1,2,\ldots$, but the function sequence (2.18) is not convergent for all large t.

(II_3) the sequence (2.18) is convergent and $\lim_{n\to\infty} q_n(t,\xi,\eta) = q(t,\xi,\eta)$, but $q(t,\xi,\eta) \notin L^2(t_o,\infty)$.

Then equation (1.4) is strongly oscillatory.

PROOF. Assume that Equation (1.1) at $\lambda=\lambda_o > 0$, has a solution $x(t) > 0$ for $t > t_o > 0$. A similar proof will hold if $x(t) < 0$ for $t > t_o$. Let $w(t)=x'(t)/\phi(x(h(t)))$. From Lemma 2.1 we can obtain (2.1). It follows that $w(t) > q_o(t,\eta_o)$, where $\eta_o=\lambda_o\varepsilon$. Hence

$$w^2(t) > q_o^2(t,\eta_o), \quad t > t_1 > t_o. \tag{2.19}$$

Suppose that (II_1) holds. If m=1 from (2.1), (2.19) implies that

$\int_t^\infty q_o^2(s,\eta_o)\,ds < \int_t^\infty w^2(s)ds < \infty$. Thus

$$q_1(t,\xi_o,\eta_o) = \xi_o\int_t^\infty q_o^2(s,\eta_o)ds + q_o(t,\eta_o) < w(t), \quad t > t_1$$

where $\xi_o = \sigma$. This is in contradiciton to the nonexistence of q_1.

If $m > 1$, from (2.1) and (2.19) we get that $q_{m-1} < w(t)$. Hence

$\int_t^\infty q_{m-1}(s,\xi_o,\eta_o)\,ds < \int_t^\infty w^2(s)\,ds < \infty$. Applying (2.1) we have that

$$q_m(t,\xi_o,\eta_o) = \xi\int_t^\infty q_{m-1}^2(s,\xi_o,\eta_o)ds + q_o(t,\eta_o) < w(t)$$

and we arrive at a contradiction of (II_1).

Suppose that (II_2) holds. From (2.18) and (2.19), we conclude that for all large t

$$q_{n-1}(t,\xi_o,\eta_o) < q_n(t,\xi_o,\eta_o) < w(t),\ n=1,2,\ldots\ . \tag{2.20}$$

Therefore $\lim_{n\to\infty} q_n(t,\xi_o,\eta_o)$ exists and has a finte limit. But this constradicts the fact that $q_n(t,\xi_\delta\,\eta_o)$ is not convergent.

Suppose that (II_3) holds. By (2.20),

$$\lim_{n\to\infty} q_n(t,\xi_o,\eta_o) = q(t,\xi_o,\eta_o) < w(t). \tag{2.21}$$

Using (2.21), we have that $\int_t^\infty q^2(s,\xi_o,\eta_o)ds < \int_t^\infty w^2(s)ds < \infty$ which contradicts the condition (II_3). This completes the proof.

THEOREM 2.4. Assume that

$$\int_{\alpha}^{\infty} \frac{du}{\phi(u)} < \infty \quad \text{and} \quad \int_{-\alpha}^{-\infty} \frac{du}{\phi(u)} < \infty, \qquad \alpha > 0. \tag{2.22}$$

Further assume that sequence (2.18) for all positive constants ξ and η satisfies any one of the following conditions:

(III_1) there is a positive integer m such that $q_n(t,\xi,\eta)$ is defined for $n=0,1,2...m$, and

$$\int_{t_o}^{\infty} q_m(s,\xi,\eta)ds = \infty;$$

(III_2) $q_n(t,\xi,\eta)$ is defined for $n=0,1,2,...$, and $\lim_{n\to\infty} q_n(t,\xi,\eta) = q(t,\xi,\eta)$ exists and satisfies

$$\int_{t_o}^{\infty} q(s,\xi,\eta)ds = \infty.$$

Then Equation (1.1) is strongly osillatory.

PROOF. Assume that Equation (1.4) at $\lambda=\lambda_o > 0$, has a nonoscillatory solution $x(t) > 0$ for $t \geq t_o$. The case $x(t) < 0$ is handled similarly. Let $w(t) = \frac{x'(t)}{\phi(x(h(t)))}$. By Lemmma 2.1, we find that (2.1) holds.

Suppose that (III_1) holds, then, as proof of Theorem 2.3,

$$q_m(t,\xi_o,\eta_o) = \xi_o \int_t^{\infty} q_m^2(s,\xi_o,\eta_o)ds + q_o(t,\xi_o,\eta_o) \leq w(t)$$

or

$$q_m(t,\xi_o,\eta_o) \leq \frac{x'(t)}{\phi(x(h(t)))} \leq \frac{x'(h(t)h'(t)}{k\phi(x(h(t)))}. \tag{2.23}$$

From (2.22) and (2.23), we have that

$$\lim_{t\to\infty} \int_{t_o}^{t} q_m(t,\xi_o,\eta_o)ds \leq \lim_{t\to\infty} \int_{x(h(t_o))}^{x(h(t))} \frac{du}{k\phi(u)} < \infty.$$

This contradicts condition (III_1).

Suppose that (III_2) holds, then it follows from (2.23) that

$$\lim_{m\to\infty} q_m(t,\xi_o,\eta_o) \leq \frac{x'(h(t)h'(t)}{k\phi(x(h(t)))}, \text{ namely } q(t,\xi_o,\eta_o) \leq \frac{x'(h(t)h'(t)}{k\phi(x(h(t)))}.$$

Hence

$$\lim_{t\to\infty} \int_{t_o}^{t} q(s,\xi_o,\eta_o)ds \leq \lim_{t\to\infty} \int_{x(h(t_o))}^{x(h(t))} \frac{du}{\phi(u)} < \infty.$$

which is again a contradiction, and the proof of the theorem is complete.

Equation (1.1) is said to be strongly bounded oscillatory if all bounded solutions of Equation (1.4) for any $\lambda\varepsilon(0,\infty)$ are oscillatory.

From the proof of Theorem (2.4), we see that the following result holds.

COROLLARY 2.5. Assume that the condition (III_1) or (III_2) holds, then Equation (1.1) is strongly bounded oscillatory.

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