RESEARCH NOTES

ON THE COEFFICIENT DOMAINS OF UNIVALENT FUNCTIONS

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ABSTRACT. Coefficient domains for functions whose derivative has positive real part in the interior of an ellipse are given in this paper.

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1. INTRODUCTION.

Let $f(z)$ be regular and satisfy the condition

$$\operatorname{Re} f'(z) > 0 \quad (1.1)$$

in a domain D . Then it is wellknown (see [1, p. 582], [2]) that $f(z)$ is univalent in D . Let D be the interior of a fixed ellipse

$E_0 = \{z = \cosh(s_0 + i\tau), 0 < \tau < 2\pi, s_0 = \tanh^{-1}(b/a), a > b > 0\}$ with foci ± 1 . Let $r_0 = a + b$ be the sum of the semi-axes of the ellipse E_0 and set $z = \cosh \eta$ where

$\eta = s + i\tau$ and $0 < s < s_0$. Then we see from the operator $\partial/\partial z = \partial/\partial \eta$ that (1.1) becomes

$$\operatorname{Re} \sqrt{\frac{2}{z-1}} f'(z) > 0 \quad (1.2)$$

for z in $\operatorname{Int}(E_0)$.

In this note we shall study coefficient domains for functions which are regular and satisfy (1.2) in $\operatorname{Int}(E_0)$. In connection with this problem see [3, Problem 6.54], [4, Problem 7.2], [5], [6], and [7] and [8, p.141].

THEOREM. Let $f(z) = \sum_{n=1}^{\infty} a_n T_n(z)$ be regular and satisfy (1.2) in $\text{Int}(E_0)$. Then for $n > 1$ we have the sharp inequalities

$$|a_n| < 2/n \sinh ns_0, \quad (1.3)$$

$$\alpha_n^2 \sinh^2 ns_0 + \beta_n^2 \cosh^2 ns_0 < 4/n^2. \quad (1.4)$$

The inequality (1.4) shows that the coefficients a_n lie in ellipses with centre at the origin and semi-axes $4/n(r_0^n \pm r_0^{-n})$ where $n=1,2,3,\dots$.

PROOF OF THE THEOREM. We see from (1.2), setting $z = \cosh(s_0 + i\tau)$, $s_0 = \log r_0$ and $a_n = \alpha_n + i\beta_n$, that

$$\text{Re}[1 + \sqrt{z^2 - 1} f'(z)] = 1 + \sum_{n=1}^{\infty} n(\alpha_n \sinh ns_0 \cos n\tau - \beta_n \cosh ns_0 \sin n\tau)$$

where $\text{Re}[1 + \sqrt{z^2 - 1} f'(z)] > 0$ in $\text{Int}(E_0)$.

Since this is a Fourier series for fixed s_0 ; we then see that

$$\int_0^{2\pi} \text{Re}[1 + \sqrt{z^2 - 1} f'(z)] d\tau = 2\pi, \quad (1.5)$$

$$\int_0^{2\pi} \text{Re}[1 + \sqrt{z^2 - 1} f'(z)] \cos n\tau = n\pi \alpha_n \sinh ns_0, \quad (1.6)$$

$$\int_0^{2\pi} \text{Re}[1 + \sqrt{z^2 - 1} f'(z)] \sin n\tau = n\pi \beta_n \cosh ns_0. \quad (1.7)$$

Using (1.5), (1.6), and (1.7) we obtain

$$\begin{aligned} |a_n| &= |\alpha_n + i\beta_n| \\ &= \left| \frac{1}{n\pi i} \int_0^{2\pi} \frac{\sinh n(s_0 + i\tau) \text{Re}[1 + \sqrt{z^2 - 1} f'(z)]}{\sinh ns_0 \cosh ns_0} d\tau \right| \\ &< \frac{1}{n\pi \sinh ns_0} \int_0^{2\pi} \text{Re}[1 + \sqrt{z^2 - 1} f'(z)] d\tau \\ &< 2/n \sinh ns_0 \end{aligned}$$

since $|\sinh n(s_0 + i\tau)| < \cosh ns_0$. This is (1.3).

We also see from (1.5), (1.6) and (1.7) that

$$\left| \alpha_n \sinh ns_0 + i\beta_n \cosh ns_0 \right| = \left| \frac{1}{n\pi i} \int_0^{2\pi} \operatorname{Re}[1+\sqrt{z^2-1} f'(z)] e^{ni\tau} d\tau \right|$$

This gives

$$< 2/n.$$

$$\alpha_n^2 \sinh^2 ns_0 + \beta_n^2 \cosh^2 ns_0 < 4/n^2$$

as required in (1.4) and the proof of the theorem is complete.

Finally we see from [9, Theorems 2 and 6] that

$$f'(z) = -1/\sqrt{z^2-1} + \int_0^{2\pi} (K(z, \bar{\xi}) / \sqrt{z^2-1}) d\psi(\tau')$$

where $\xi = \cosh(s_0 + i\tau')$, $z = \cosh(s + i\tau)$, $0 < s < s_0$, $0 < \tau' < 2\pi$ and

$0 < \tau < 2\pi$ plays the role of the external function in this case.

REMARK. Normalizing in the sense of [3, Remark 2] we obtain the analogous results in [2, Theorem 1].

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