SUBLINEAR FUNCTIONALS AND KNOPP'S CORE THEOREM

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ABSTRACT. In this paper we are concerned with inequalities involving certain sublinear functionals on m , the space of real bounded sequences. Such inequalities being analogues of Knopp's Core theorem.

KEY WORDS AND PHRASES. Core theorem, Sublinear functionals, Infinite matrices, Almost convergence.

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1. INTRODUCTION.

Let m be the linear space of real bounded sequences with the usual supremum norm. We write

$$m_0 = \{x \in m : \sup_n \sum_{k=0}^n x_k < \infty\}$$

Let $\mathcal{A}$ be the sequence of infinite matrices $(A^i) = (a_{nk}(i))$. Given a sequence $x = (x_k)$ we write

$$A_n^i(x) = \sum_{k=0}^{\infty} a_{nk}(i) x_k \quad (1.1)$$

if it exists for each n and $i \geq 0$. We also write Ax for $(A_n^i(x))_{i,n=0}^{\infty}$. The sequence $x = (x_k)$ is said to be summable to the value s by the method $(\mathcal{A})$ if

$$A_n^i(x) \rightarrow s \quad (n \rightarrow \infty, \text{ uniformly in } i) \quad (1.2)$$

If (1.2) holds, then we write $x \rightarrow s(\mathcal{A})$.

If we define $a_{nk}(i)$ by

$$a_{nk}(i) = \begin{cases} 1/n+1 & , \quad i \leq k \leq i+n \\ 0 & , \quad \text{otherwise} \end{cases}$$

then $(\mathcal{A})$ reduces to the method f (Lorentz [1]). In the case

$$a_{nk}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk}$$

$(\mathcal{A})$ reduces to the almost summability method (King [2]). If $\mathcal{A} = A = (a_{nk})$, then we get the usual summability method (A) .

The method $(\mathcal{A})$ is said to be conservative if $x \rightarrow s$ implies $x \rightarrow s'(\mathcal{A})$. If $(\mathcal{A})$ is conservative and $s = s'$, then $(\mathcal{A})$ is called regular.

It is well known, (Stieglitz [3]), that $(\mathcal{A})$ is regular if and only if the following conditions hold:

$$\sum_k |a_{nk}(i)| < \infty, \quad (\text{for all } n, \text{ for all } i), \quad (1.3)$$

and there exist an integer m such that

$$\sup_{i \geq 0, n \geq m} \sum_k |a_{nk}(i)| < \infty \quad (1.4)$$

$$\lim_n a_{nk}(i) = 0, \quad \text{uniformly in } i, \quad (1.5)$$

$$\lim_n \sum_k a_{nk}(i) = 1, \quad \text{uniformly in } i, \quad (1.6)$$

Throughout the paper we write

$$\|\mathcal{A}\| = \sup_{n,i} \sum_k |a_{nk}(i)| < \infty \quad (1.7)$$

to mean that, there exists a constant M such that

$$\sum_k |a_{nk}(i)| \leq M \quad (\text{for all } n, \text{ for all } i) \quad (1.8)$$

and the series

$$\sum_k a_{nk}(i) \quad (1.9)$$

converges uniformly in i for each n .

If, for every bounded sequence x , $x \rightarrow s(\mathcal{A})$ then $(\mathcal{A})$ is said to be a Schur method

Throughout the paper we consider only real matrices and real bounded sequences.

In this paper we are concerned with inequalities involving certain sublinear functionals on m , the space of real bounded sequences. Such inequalities being analogues of Knopp's Core theorem. That theorem determines a class of regular matrices for which

$$\limsup Ax \leq \limsup x$$

for all $x \in m$, see e.g Cooke [4], Maddox [5], Simons [6]. This result has also been extended to coregular matrices by Rhoades [7], Schaefer [8], and, Das [9].

Before stating the theorems to be proved, we introduce some further notation.

$$\mathcal{L}(x) = \liminf x_n; \quad L(x) = \limsup x_n, \quad \|x\| = \sup |x_n|$$

$$\mathcal{L}^*(x) = \liminf_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r$$

$$L^*(x) = \limsup_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r$$

$$W^*(x) = \inf_{z \in m_0} L^*(x+z)$$

If f, g are any two of the above functionals, we shall write $fA \leq gB$ to denote that, for every bounded sequence x , the transforms Ax and Bx are defined and bounded and $f(Ax) \leq g(Bx)$.

2. THE MAIN RESULTS.

We write, for $x \in m$,

$$Q_A(x) = \limsup_n \sup_i \sum_k a_{nk}(i) x_k$$

and

$$q_A(x) = \liminf_n \sup_i \sum_k a_{nk}(i) x_k$$

With this notation we have

THEOREM 1. Let $\|A\| < \infty$. Then

$$Q_A \leq L \quad (2.1)$$

if and only if (A) is regular and

$$\sum_k |a_{nk}(i)| \rightarrow 1 \quad (n \rightarrow \infty, \text{ uniformly in } i) \quad (2.2)$$

PROOF. Necessity. Let $x = (x_k)$ be a convergent sequence. Then $f(x) = L(x) = \lim x$. By (2.1), we have

$$f(x) \leq -Q_A(-x) \leq Q_A(x) \leq L(x).$$

Hence we get that $Q_A(x) = q_A(x) = \lim x$. So (A) is regular.

Since (A) is regular, the requirement of Lemma 2, (Das [9]), is satisfied. Hence there exists $y \in m$ such that $\|y\| \leq 1$ and

$$Q_A(y) = \limsup_n \sup_i \sum_k |a_{nk}(i)|.$$

Hence, taking $x = e = (1, 1, \dots)$, we have

$$\begin{aligned} 1 = q_A(e) &\leq \liminf_n \sup_i \sum_k |a_{nk}(i)| \\ &\leq \limsup_n \sup_i \sum_k |a_{nk}(i)| = Q_A(y) \leq L(y) \leq \|y\| \leq 1 \end{aligned}$$

which proves the necessity of (2.2)

Sufficiency. We define, for any real λ , $\lambda^+ = \max(\lambda, 0)$, $\lambda^- = \max(-\lambda, 0)$. Then $|\lambda| = \lambda^+ + \lambda^-$ and $\lambda = \lambda^+ - \lambda^-$. Hence

$$\sum_k a_{nk}(i) x_k = \sum_{k < m} a_{nk}(i) x_k + \sum_{k \geq m} (a_{nk}^+(i)) x_k - \sum_{k \geq m} (a_{nk}^-(i)) x_k$$

So we have

$$\sum_k a_{nk}(i) x_k \leq \|x\| \sum_{k < m} |a_{nk}(i)| + (\sup_k x_k) \sum_{k \geq m} |a_{nk}(i)| + \|x\| \sum_{k \geq m} (|a_{nk}(i)| - a_{nk}(i))$$

By hypothesis, we get that $Q_A(x) \leq L(x)$.

REMARK. We could use Theorem 2, (Das [9]), to get the sufficiency.

COROLLARY.2. We have on m ,

$$L \leq L^* \leq L_*^* \leq L_*$$

PROOF. In theorem 1, it is enough to take

$$a_{nk}(i) = \begin{cases} 1/n+1 & , \quad i \leq k \leq i+n \\ 0 & , \quad \text{otherwise} \end{cases}$$

We deduce at once from Corollary 2 that if a sequence x is convergent to s , then it is almost convergent to s which is a well-known result.

We note that, by considering Theorem 1, one may get necessary and sufficient conditions for $L_*^*A \leq L$ and $LA \leq L_*$.

In the next theorem we consider the inequality $LA \leq L^*$.

THEOREM.3. $LA \leq L^*$ if and only if A is strongly regular and

$$\sum_k |a_{nk}| \rightarrow 1 \quad (n \rightarrow \infty) \quad (2.3)$$

PROOF. Recall that a matrix A is called strongly regular if it maps all almost convergent sequences into the convergent sequences and $\lim Ax = f - \lim x$.

We first prove the necessity. It is easy to see that $L^* \leq LA \leq L_*^* \leq L_*$. If x is almost convergent then $f - \lim x = L^*(x) = L_*(x)$. Hence, by the hypothesis, $L(Ax) = L(Lx) = f - \lim x$. So A is strongly regular. Using the fact that $L^* \leq L$ (see Corollary 2) and that $LA \leq L^*$, we get that $LA \leq L_*$. Now the necessity of (2.3) follows from Knopp's Core theorem (see, e.g. Maddox [5]).

We note in passing that a matrix A is strongly regular, Lorentz [1], if and only if it is regular and that

$$\sum_k |a_{nk} - a_{n,k+1}| \rightarrow 0 \quad (n \rightarrow \infty) \quad (2.4)$$

Sufficiency. Given $\varepsilon > 0$, we can find a positive integer p such that for $x \in m$ and for all $k \geq 0$,

$$\frac{1}{p+1} \sum_{r=k}^{k+p} x_r < L^*(x) + \varepsilon \quad (2.5)$$

(We fix p throughout the analysis).

As in Lorentz's proof (see [1]; Th. 7) one can show that

$$\begin{aligned} \sum_{k=0}^{\infty} a_{nk} x_k &= \sum_{k=0}^{\infty} a_{nk} \frac{1}{p+1} \sum_{r=k}^{k+p} x_r \\ &= \sum_{k=p}^{\infty} \left(\frac{a_{nk} + \dots + a_{n,k-p}}{p+1} - a_{nk} \right) \\ &\quad + \sum_{k=0}^{p-1} a_{nk} x_k \\ &\quad + \sum_{k=0}^{p-1} \left(\frac{a_{nk} + \dots + a_{n,k-p+1}}{p+1} \right) x_k \end{aligned} \quad (2.6)$$

Since $x \in m$, it follows from the regularity of A that the third and fourth sigmas in (2.6) tend to zero as $n \rightarrow \infty$. If we write

$$F_{np} = \sum_{k=p}^{\infty} \left(\frac{a_{nk} + \dots + a_{n,k-p}}{p+1} - a_{nk} \right) x_k$$

then

$$\begin{aligned} |F_{np}| &\leq \frac{1}{p+1} \sum_{k=p}^{\infty} |a_{nk} + \dots + a_{n,k-p} - (p+1)a_{nk}| |x_k| \\ &\leq \frac{\|x\|}{p+1} \sum_{r=0}^p \sum_{k=p}^{\infty} |a_{n,k-r} - a_{nk}| \\ &\leq \frac{\|x\|}{p+1} \sum_{r=0}^p r \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| \\ &\leq \frac{p}{2} \|x\| \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| \end{aligned} \quad (2.7)$$

Since A is strongly regular, (2.4) holds. Thus the expression in (2.7) tends to zero as $n \rightarrow \infty$. Hence we find that

$$\begin{aligned} L(Ax) &\leq \limsup_n \sum_{k=0}^{\infty} a_{nk} \left(\frac{x_k + \dots + x_{k+p}}{p+1} \right) \\ &\leq \limsup_n \sum_k \left(a_{nk}^+ \right) \left(\frac{x_k + \dots + x_{k+p}}{p+1} \right) \\ &\quad - \limsup_n \sum_k \left(a_{nk}^- \right) \left(\frac{x_k + \dots + x_{k+p}}{p+1} \right) \end{aligned}$$

By (2.5), we have

$$L(Ax) \leq (L^*(x) + \epsilon) \limsup_n \sum_k |a_{nk}| + \|x\| \limsup_n \sum_k (|a_{nk}| - a_{nk})$$

Using the regularity of A and (2.3) we get that

$$L(Ax) \leq L^*(x) + \epsilon.$$

Since ϵ is arbitrary, sufficiency follows.

THEOREM.4. $L^*A \leq L^*$ if and only if A is F -regular and

$$\limsup_n \sum_i \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} = 1 \quad (2.8)$$

PROOF. Recall that A is called F -regular if it maps F , the class of all almost convergent sequences, into itself and $f\text{-}\lim Ax = f\text{-}\lim x$. Corollary to Theorem 4 in [10] gives the necessary and sufficient conditions for A to be F -regular.

We now come to the proof of necessity.

One can easily show that

$$\mathbb{L}^*(x) \leq \mathbb{L}^*(Ax) \leq L^*(Ax) \leq L^*(x).$$

If $x \in F$, then $\mathbb{L}^*(x) = L^*(x) = f - \lim x$. Hence $\mathbb{L}^*(Ax) = L^*(Ax) = f - \lim x$. So A is F -regular.

To get the necessity of (2.8), we define $(b_{nk}(i))$ by

$$b_{nk}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk}$$

Observe now that the conditions of Lemma 2, Das [9], are satisfied. So we must have a bounded sequence y such that $\|y\| \leq 1$ and

$$Q_B(y) = \limsup_n \sup_i \sum_k |b_{nk}(i)| \quad (2.9)$$

Hence by (2.9) and F -regularity of A , we get

$$\begin{aligned} 1 &\leq \liminf_n \sup_i \sum_k \left| \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right| \\ &\leq \limsup_n \sup_i \sum_k \left| \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right| \\ &= \limsup_n \sup_i \sum_k \left(\frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right) y_k \leq L^*(y) \leq \|y\| \leq 1 \end{aligned}$$

which proves (2.8).

Sufficiency. We first note that

$$\begin{aligned} L^*(Ax) &= \limsup_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} A_r(x) \\ &= \limsup_n \sup_i \sum_k \left(\frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right) x_k \end{aligned}$$

If we set

$$b_{nk}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk}$$

then (2.6) with a_{nk} replaced by $b_{nk}(i)$ holds. Since $x \in m$ and A is F -regular, Corollary to Theorem 4 in [10] yields that the second and third sigmas with a_{nk} replaced by $b_{nk}(i)$, tend to zero as $n \rightarrow \infty$, uniformly in i . On the other hand $\|F_{np}\|$, with a_{nk} replaced by $b_{nk}(i)$ is not greater than

$$\frac{p}{2} \|x\| \sum_k |b_{nk}(i) - b_{n,k+1}(i)| = \frac{p}{2} \|x\| \sum_k \left| \frac{1}{n+1} \sum_{r=i}^{i+n} (a_{rk} - a_{r,k+1}) \right|$$

Since A is F -regular, the last sigma tends to zero as $n \rightarrow \infty$, uniformly in i . Hence we have, by (2.5), that

$$L^*(Ax) \leq \limsup_n \sup_i \sum_k \left(\frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right) \left(\frac{x_k + \dots + x_{k+p}}{p+1} \right)$$

$$\leq (L^*(x) + \epsilon) \limsup_n \sup_i \sum_k \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} + \|x\| \limsup_n \sup_i \sum_k \left(\left(1 - \frac{1}{n+1}\right) \sum_{r=i}^{i+n} a_{rk} - \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} \right)$$

Using (2.8) and the fact that Λ is F-regular, we get

$$L^*(\Lambda x) \leq L^*(x) + \epsilon.$$

Since ϵ is arbitrary, the required conclusion follows.

We now give another inequality sharper than that of Theorem 4. (See Theorem 6 below). It is also an analogue of Theorem 3 given by Devi [11]. We first need to prove a Lemma.

LEMMA.5. Let $(B^{(i)})$ be a sequence of infinite matrices such that (1.3) and (1.5), with $a_{nk}(i)$ replaced by $b_{nk}(i)$, hold. Then, for every $z \in m_0$, we have $\mathbf{B} z = \mathbf{D} y$, where

$$\mathbf{D} = (d_{nk}(i)) = (b_{nk}(i) - b_{n,k+1}(i)),$$

and

$$y = (y_n) = \left(\sum_{k=1}^n z_k \right) \in m.$$

If, further

$$\lim_n \sum_k |d_{nk}(i)| = 0, \quad \text{uniformly in } i,$$

then $y \rightarrow O(\mathbf{D})$ and $z \rightarrow O(\mathbf{B})$.

PROOF. The first assertion follows from Abel's partial summation. The second one is a consequence of the Result (3.2.1) given by Duran [10].

We are now in a position to give the inequality mentioned above.

THEOREM.6. $L^*A \leq W^*$ if and only if A is F-regular and (2.8) holds.

Before proving the theorem we note that W^* is well-defined (see Devi [11]).

We now come to the proof.

Suppose that $L^*A \leq W^*$. Since $W^* \leq L^*$, it follows from Theorem 4 that A is F-regular and that (2.8) holds.

Conversely suppose that A is F-regular and (2.8) holds. By Theorem 4, we get

$$u(x) = \inf_{z \in m_0} L^*(\Lambda(x+z)) \leq W^*(x) \quad (2.10)$$

On the other hand

$$L^*(\Lambda(x+z)) = \limsup_n \sup_i \sum_k \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk} (x_k + z_k) \quad (2.11)$$

Now write

$$b_{nk}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} a_{rk}.$$

Since A is F -regular, the requirement of Lemma 5 is satisfied. Hence we have that $\gamma \rightarrow \gamma(0)(B)$. So, we get

$$u(x) \geq \inf_{z \in m_0} \left\{ L^*(\Lambda x) + L^*(\Lambda z) \right\} = L^*(\Lambda x) \quad (2.12)$$

Hence the required conclusion follows from (2.10) and (2.11).

The following theorem is a generalization of Result VII given by Kuttner and Maddox [12].

THEOREM.7. Let $\|A\| < \infty$ and $\|B\| < \infty$. Then $Q_A(x) \leq q_B(x)$ if and only if (B) is a Schur method and

$$\sum_k |a_{nk}(i) - b_{nk}(i)| \rightarrow 0 \quad (n \rightarrow \infty, \text{ uniformly in } i)$$

PROOF. Since the proof uses the technique that Kuttner and Maddox used, [12], we omit the details.

We conclude the paper with the following remark: Since no Schur method is regular, Theorem 7 includes the result that $Q_A(x) \leq q_B(x)$ is impossible when (B) is a regular method. For example,

$$Q_A(x) \leq L(x) \quad (\text{for every } x \in m),$$

is impossible.

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