A NOTE ON THE k -DOMINATION NUMBER OF A GRAPH

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ABSTRACT. The k -domination number of a graph $G = G(V, E)$, $\gamma_k(G)$, is the least cardinality of a set $X \subset V$ such that any vertex in $V \setminus X$ is adjacent to at least k vertices of X .

Extending a result of Cockayne, Gamble and Shepherd [4], we prove that if $\delta(G) > \frac{n+1}{n} k - 1$, $n \geq 1, k \geq 1$ then, $\gamma_k(G) < \frac{np}{n+1}$, where p is the order of G .

KEY WORDS AND PHRASES. k -dominating set and k -domination Number.

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1. INTRODUCTION.

A set X of vertices of a graph $G = G(V, E)$ is k -dominating if each vertex of $V \setminus X$ is adjacent to at least k vertices of X . The k -domination number of a graph G , $\gamma_k(G)$, is the smallest cardinality of a k -dominating set of G .

We write $\delta = \delta(G)$ for the minimum degree of vertices in G and $|G| = p$ is the number of vertices of G .

Several results concerning $\gamma_k(G)$ have been established by Fink and Jacobson [1], [2] who showed that $\gamma_k > \frac{kp}{\Delta+k}$, and recently by Favaron [3].

As for the upper bound, Cockayne, Gamble and Shepherd proved the following:

THEOREM 1.1. If G has p vertices and $\delta \geq k$, then $\gamma_k(G) < \frac{kp}{k+1}$.

2. MAIN RESULTS.

Our aim in this note is to extend Theorem 1.1 and give a shorter proof of that given in Cockayne, Gamble, and Shepherd [4]. We prove,

THEOREM 2.1. Let n, k be positive integers and G a graph such that

$$\delta(G) \geq \frac{n+1}{n} k - 1. \text{ Then, } \gamma_k(G) \leq \frac{np}{n+1}.$$

PROOF. Let $V_1, V_2, \dots, V_{n+1}$ be a partition of $V(G)$ into $n+1$ subsets which maximizes

the number of edges in E' where $E' = E(G) \setminus \bigcup_{i=1}^{n+1} E(\langle V_i \rangle)$ and $\langle V_i \rangle$ is the subgraph induced on the vertex set V_i .

By a classical argument of Erdős [5] we have that for every $x \in V$, $\deg_H(x) \geq \lfloor \frac{n}{n+1} \deg_G(x) \rfloor$, where $H = H(V', E')$, $V' = V$, and E' is as above. Hence we conclude that:

$$\deg_H(x) \geq \lfloor \frac{n}{n+1} (\frac{n+1}{n} k - 1) \rfloor = \lfloor k - \frac{n}{n+1} \rfloor = k.$$

Assume W.L.O.G. that $|V_1| \geq |V_2| \geq \dots \geq |V_{n+1}|$. Then the set $A = \bigcup_{i=2}^{n+1} V_i$ is a k -

dominating set of G since each vertex $x \in V_1$ is adjacent to at least k vertices of A . Thus it follows that $\gamma_k(G) \leq p - |V_1| \leq \frac{np}{n+1}$.

COROLLARY 1. [4] If $\delta(G) \geq k$ then $\gamma_k(G) \leq \frac{kp}{k+1}$.

PROOF. Take $n = k$ in Theorem 2.1.

COROLLARY 2: If $\delta(G) \geq 2k - 1$ then $\gamma_k(G) \leq \frac{p}{2}$.

REMARK. Using a similar argument we can prove the following:

$$\text{If } \delta(G) \geq k \geq 1 \text{ and } \chi(G) = n, \text{ then } \gamma_k(G) \leq \frac{(n-1)p}{n}.$$

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