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THE STRUCTURE OF HOMOMORPHISMS FROM BANACH ALGEBRAS OF DIFFERENTIABLE FUNCTIONS INTO FINITE DIMENSIONAL BANACH ALGEBRAS

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ABSTRACT: We show that the structure of continuous and discontinuous homomorphisms from the Banach algebra $C^n[0,1]$ of n times continuously differentiable functions on the unit interval [0,1] into finite dimensional Banach algebras is completely determined by higher point derivations.

KEY WORDS AND PHRASES. Banach algebras, homomorphisms, local algebras, singularity set, higher point derivations.
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0. Introduction.

It is well known that the Banach algebra $C^n[0,1]$ is generated by $z(t)=t$, $0\leq t\leq 1$. Thus a continuous homomorphism ν of $C^n[0,1]$ into a Banach algebra $\mathfrak{B}$ is completely determined by $\nu(z)$. We are mainly interested in the structure of discontinuous homomorphisms ν from $C^n[0,1]$ into finite dimensional Banach algebras. In 1980 Bade, Curtis and Laursen [1] showed that these homomorphisms have a striking degree of continuity: the restriction of ν to $C^{2n}[0,1]$ is continuous with respect to the C^{2n}-norm. So, if we can obtain an explicit structure of continuous homomorphism ν from $C^n[0,1]$ into finite dimensional Banach algebras we may understand the behavior of discontinuous ones; that will be our approach to this problem.

1. Preliminaries.

Let $C^n[0,1]$ denote the algebra of all complex valued functions on [0,1] which have n continuous derivatives. It is well known that $C^n[0,1]$ is a Banach algebra under the norm

$$\|f\| = \max_{t\in[0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!}$$

whose structure space is [0,1]. We will need a characterization of the square of the closed primary ideals with finite codimension in $C^n[0,1]$. We use the notation

$$M_{n,k}(t_0) = \{f\in C^n[0,1] \mid f^{(j)}(t_0) = 0;\ j = 0,1,\dots,k\}.$$

These are precisely the closed ideals of finite codimension contained in the maximal ideal $M_{n,k}(t_0)$ of functions vanishing at t_0. Writing $M_{n,k}$ for $M_{n,k}(0)$ and setting $z(t) = t$, $0\leq t\leq 1$, we have:

1.1 THEOREM. Let n be a positive integer. Then

(i) $M_{n,0}^2 = zM_{n,0} = \{f | f(0) = f'(0) = 0$ and $f^{(n+1)}(0)$ exists$\}$,

(ii) $M_{n,k}^2 = z^{k+1}M_{n,k}$, $1 \leq k \leq n-1$,

(iii) $M_{n,n}^2 = z^n M_{n,n}$.

Part (i) is from [2, Example 3]. Part (ii) is due to Dales and McClure [3,Theorem 3.1]. The proof of part (iii) can be found in [4].

The squares of the closed primary ideals $M_{n,k}(t_0)$ at other points t_0 in [0,1] are given exactly by similar formulas, where z is replaced by $z-t_0$. We also need the following concepts and facts in automatic continuity theory.

1.2 DEFINITION. If T: $\mathcal{A} \to \mathcal{B}$ is a linear map and $\mathcal{A}$, $\mathcal{B}$ are Banach spaces, then the separating space of T, $\mathcal{S}(T)$ is defined by

$$\mathcal{S}(T) = \{y \in \mathcal{B} | \ \exists \{x_n\} \subset \mathcal{A},\ x_n \Rightarrow 0, \text{ and } T(x_n) \Rightarrow y\}.$$

This space measures the discontinuity of T because $\mathcal{S}(T) = \{0\}$ if and only if T is discontinuous, by the closed graph theorem. More detailed discussion on $\mathcal{S}(T)$ can be found in [5].

1.3 DEFINITION. If $\mathcal{A}$, $\mathcal{B}$ are Banach algebras, and T: $\mathcal{A} \to \mathcal{B}$ is a homomorphism with separating space $\mathcal{S}(T)$, then the continuity ideal of T, $\mathcal{I}(T)$ is defined by

$$\mathcal{I}(T) = \{x \in \mathcal{A} | \ T(x)\mathcal{S}(T) = (0)\}.$$

Let $\mathcal{B}$ be a Banach algebra and ν: $C^n[0,1] \to \mathcal{B}$ be a homomorphism. It is shown in [6,7] that the continuity ideal $\mathcal{I}(\nu)$ has finite hull and contains the ideal J(F) of all functions vanishing in neighborhoods of F = hull($\mathcal{I}(\nu)$). F is called the singularity set of ν.

1.4 THEOREM. Let n be a positive integer and ν: $C^n[0,1] \to \mathcal{B}$ be a discontinuous homomorphism with singularity set F = {0}. Consider the following statements:

(a) $\mathcal{S}(\nu)$ is finite dimensional,

(b) $\mathcal{I}(\nu)$ has finte codimension,

(c) $\mathcal{I}(\nu)$ is closed and contains $M_{n,n-1}$,

(d) $\mathcal{S}(\nu)^2 = \{0\}$,

(e) $z^n \in \mathcal{I}(\nu)$,

(f) ν is continuous on $M_{n,n}^2 = z^n M_{n,n}$ for the graph norm $|||f||| = ||f|| + ||\frac{f}{z^n}||$,

(g) ν is C^{2n}-continuous (i.e. the restriction of ν to C^{2n} is continuous with respect to the C^{2n}-norm).

We have the following implications:

$$\text{(a)} \Rightarrow \text{(b)} \Leftrightarrow \text{(c)} \Leftrightarrow \text{(d)} \Leftrightarrow \text{(e)} \Leftrightarrow \text{(f)} \Rightarrow \text{(g)}.$$

For the proof see [1].

2. Algebraic results.

Let ν be a homomorphism of $C^n[0,1]$ into a finite dimensional Banach algebra $\mathcal{B}$. We may assume that ν is onto by considering ν: $C^n[0,1] \to \nu(C^n[0,1])$. We shall reduce the study of ν to the case where the range space is a finite dimensional local algebra.

Let $\mathfrak{B}$ be the range of ν. Since $\mathfrak{B}$ is a finite dimensional commutative algebra with a unit e, the Wedderburn principal theorem states that $\mathfrak{B} = \mathcal{A} + \mathfrak{R}$, where $\mathfrak{R}$ is the radical of $\mathfrak{B}$. Now $\mathcal{A}$ is a semisimple commutative algebra with unit. By the Wedderburn structure theorem for finite dimensional algebras we can write

$$\mathcal{A} = e_1\mathcal{A} \oplus \ldots. \oplus e_m\mathcal{A},$$

where

$$e_i\, e_j = 0, \qquad i \neq j,$$
$$e_i^{\,2} = e_i\ , \qquad i = 1,2,\ldots, m\ ,$$
$$e = e_1 + \ldots + e_m\ ,$$

and $e_i\mathcal{A}$ is a simple commutative algebra with unit e_i , so that $e_i\mathcal{A} \approx \mathbf{C}$. Thus we can write

$$\mathfrak{B} = e_1\mathfrak{B} \oplus \ldots \oplus e_m\mathfrak{B},$$

where each $e_i\mathfrak{B}$ is a local algebra which may be isomorphic to $\mathbf{C}$. Moreover $\nu = e_1\nu + \ldots + e_m\nu$ and each $e_i\nu$ is a homomorphism of $C^n[0,1]$ onto $e_i\mathfrak{B}$. If $e_i\mathfrak{B} \approx \mathbf{C}$ then $e_i\nu$ is just a multiplicative linear functional and is of the form $e_i\nu(f) = f(t_0)\, e_i$ for some t_0 in $[0,1]$. It remains to consider the case when $e_i\mathfrak{B}$ is a local algebra which is not isomorphic to $\mathbf{C}$. Our next objective is to characterize the kernel of ν.

2.1 LEMMA. Let ν be a homomorphism of $C^n[0,1]$ onto a finite dimensional algebra. If $t_0 \in$ hull(ker ν) then

(i) $(z - t_0)^m \in \ker \nu$ for some positive integer m,

(ii) The ideal $\mathrm{J}(t_0)$ of all functions vanishing in neighborhoods of t_0 is contained in $\ker \nu$.

PROOF: (i) Suppose that $\nu(z-t_0)$ is not nilpotent. Then $\nu(z-t_0)$ is invertible so that there exists g in $C^n[0,1]$ such that $\nu(z-t_0)\nu(g) = \nu(1)$. Thus $(z-t_0)g = 1 + f$ for some f in $\ker\nu$. But $(z-t_0)g \in \mathrm{M}_{n,0}(t_0)$ and $f \in \ker\nu \subset \mathrm{M}_{n,0}(t_0)$ so that $1 \in \mathrm{M}_{n,0}(t_0)$. This is a contradiction.

(ii) Let $f \in \mathrm{J}(t_0)$. Choose $g \in \mathrm{J}(t_0)$ such that g is identically one on the support of f. We claim that $\nu(g)$ is nilpotent. Suppose not, then $\nu(g)$ is invertible and $e = \nu(h)$ for some $h \in \mathrm{J}(t_0)$. So $1 = h + v$ for some $v \in \ker\nu$. This is a contradiction since $h \in \mathrm{J}(t_0) \subset \mathrm{M}_{n,0}(t_0)$ and $v \in \mathrm{M}_{n,0}(t_0)$. Thus $g^m \in \ker\nu$ for some m and we have $f = fg^m \in \ker\nu$.

Immediately following this lemma we have:

2.2 COROLLARY. Let ν be a homomorphism of $C^n[0,1]$ onto a finite dimensional local algebra. The hull of $\ker\nu$ consists of exactly one point t_0 and therefore the singularity set $\mathrm{F} = \mathrm{hull}(\mathfrak{I}(\nu)) \subseteq \{t_0\}$.

PROOF: Let t_0 and t_1 be in hull($\ker\nu$). By 2.1 there exist positive integers m_0 and m_1 such that $\nu(z-t_0)^{m_0} = 0$ and $\nu(z-t_1)^{m_1} = 0$. Then $\nu[(z-t_0) - (z-t_1)]^{m_0+m_1} = 0$ so that $t_1 - t_0 = 0$. Thus $\mathrm{hull}(\ker\nu) = t_0$ for some t_0 in $[0,1]$. Since $\ker\nu \subseteq \mathfrak{I}(\nu)$ it follows that $\mathrm{hull}(\mathfrak{I}(\nu)) \subseteq \{t_0\}$.

Without loss of generality we shall take $\mathrm{hull}(\ker\nu)$ to be $\{0\}$. With this assumption we now describe ν for the case when it is continuous.

2.3 THEOREM. Let ν be a continuous homomorphism of $C^n[0,1]$ onto a finite dimensional local algebra with $\operatorname{hull}(\ker\nu) = \{0\}$. There exists a positive integer $k \leq n+1$ such that

$$\nu(f) = \sum_{i=1}^{k-1} \delta_i(f)\nu(z)^i ,$$

where $\delta_i(f) = \dfrac{f^{(i)}(0)}{i!}$.

PROOF: Since ν is continuous, $\ker\nu$ is a closed primary ideal of finite codimension. Thus $\ker\nu = \mathrm{M}_{n,k-1}$ for some $k \leq n+1$. Let $f \in C^n[0,1]$, we write $f = \sum_{i=1}^{k-1} \delta_i(f)z^i + Rf$, where $Rf \in \mathrm{M}_{n,k-1} = \ker\nu$. Then

$$\nu(f) = \sum_{i=0}^{k-1} \delta_i(f)\nu(z)^i .$$

The sequence of linear functionals $\delta_1, \dots, \delta_{k-1}$ is called a continuous higher point derivation of order $k-1$ on $C^n[0,1]$ at δ_0. We refer to [3] for a complete description of the order and continuity properties of higher point derivations on $C^n[0,1]$.

3. The structure of discontinuous homomorphisms of $C^n[0,1]$ onto finite dimensional local algebras.

We now turn our attention to discontinuous homomorphisms ν of $C^n[0,1]$ onto a finite dimensional local algebra $\mathfrak{B}$ with $\operatorname{hull}(\ker\nu) = \{0\}$. First we characterize $\ker\nu$.

3.1 LEMMA. $\overline{\ker\nu} = \nu^{-1}(\mathfrak{I}(\nu))$.

PROOF: Let $f \in \overline{\ker\nu}$, there exists $\{f_m\} \subset \ker\nu$ with $f_m \Rightarrow f$. Then $f - f_m \Rightarrow 0$ and $\nu(f-f_m) \Rightarrow \nu(f)$ so that $\nu(f) \in \mathfrak{I}(\nu)$. Hence $\overline{\ker\nu} \subseteq \nu^{-1}(\mathfrak{I}(\nu))$.
Now let $f \in \nu^{-1}(\mathfrak{I}(\nu))$. By definition of $\mathfrak{I}(\nu)$, there exists $\{f_k\} \subset C^n[0,1]$ with $f_k \Rightarrow 0$ and $\nu(f_k) \Rightarrow \nu(f)$. Since $\overline{\ker\nu}$ has finite codimension in $C^n[0,1]$, there exists a subspace V with $\dim \mathrm{V} < \infty$ such that $C^n[0,1] = \overline{\ker\nu} \oplus \mathrm{V}$. So we can write $f_k = g_k + v_k$ where $g_k \in \overline{\ker\nu}$ and $v_k \in \mathrm{V}$. But $f_k \Rightarrow 0$ so that $g_k \Rightarrow 0$ and $v_k \Rightarrow 0$. Since $\dim\mathrm{V} < \infty$, $\nu(v_k) \Rightarrow 0$ so that $\nu(g_k) = \nu(f_k) - \nu(v_k) \Rightarrow \nu(f)$. Again we can write $\overline{\ker\nu} = \ker\nu \oplus \mathrm{W}$, where $\dim \mathrm{W} < \infty$, so $g_k = h_k + w_k$ where $h_k \in \ker\nu$ and $w_k \in \mathrm{W}$. Then $\nu(g_k) = \nu(w_k) \Rightarrow \nu(f)$ so that $\nu(f) \in \nu(\mathrm{W})$. Thus $f \in \mathrm{W} + \ker\nu = \overline{\ker\nu}$ and we conclude that $\nu^{-1}(\mathfrak{I}(\nu)) \subseteq \overline{\ker\nu}$.

3.2 LEMMA. Let k be the integer for which $\overline{\ker\nu} = \mathrm{M}_{n,k-1}$, $k \leq n+1$. Then $\mathrm{M}^2_{n,k-1} \subseteq \ker\nu$ and

$$\mathfrak{B} = \operatorname{span}\{ e, \nu(z), \dots, \nu(z)^{k-1}\} \oplus \mathfrak{I}(\nu).$$

PROOF: The first statement is clear since $\overline{\ker\nu}$ is a closed primary ideal of finite codimension. By 3.1 $\nu(\mathrm{M}_{n,k-1}) = \mathfrak{I}(\nu)$. Let $f, g \in \mathrm{M}_{n,k-1}$, then $\nu(fg) = \nu(f)\nu(g) \in \mathfrak{I}(\nu)^2 = \{0\}$ by 1.4. Since

$$C^n[0,1] = \operatorname{span}\{1, z, \dots, z^{k-1}\} \oplus \overline{\ker\nu}$$

we have

$$\mathfrak{B} = \nu(C^n[0,1]) = \operatorname{span}\{e, \nu(z), \dots, \nu(z)^{k-1}\} + \mathfrak{I}(\nu)$$

by 3.1. To see that the sum is direct let

$$b = a_0 e + a_1\nu(z) + \dots + a_{k-1}\nu(z)^{k-1} \in \mathfrak{I}(\nu)$$

and suppose $b \neq 0$. Let j be the smallest integer such that $a_j \neq 0$, then

$$\nu(z)^j\{\sum_{i=j}^{k-1} a_i\nu(z)^{i-j}\} \in \mathfrak{I}(\nu).$$

But $\sum_{i=j}^{k-1} a_i\nu(z)^{i-j}$ is invertible since $a_j \neq 0$, so $\nu(z)^j \in \mathfrak{I}(\nu)$. By 3.1 $z^j \in \overline{\ker\nu} = M_{n,k-1}$. This is a contradiction since $j \leq k-1$.

We are now in position to describe discontinuous homomorphisms.

3.3 THEOREM. Let ν be a discontinuous homomorphism of $C^n[0,1]$ onto a finite dimensional local algebra $\mathfrak{B}$ with hull$(\ker\nu) = \{0\}$ and $\overline{\ker\nu} = M_{n,k-1}$. There exist $b_1, \ldots, b_m$ in $\mathfrak{I}(\nu)$ and discontinuous linear functionals $\gamma_1, \ldots, \gamma_m$ on $C^n[0,1]$ which vanish on polynomials and on the principal ideal $z^k C^n[0,1]$ such that

$$\nu(f) = \sum_{l=0}^{k+k_1} d_l(f)\nu(z)^l + \sum_{l=0}^{i_1} \gamma_1(z^{i_1-l}f)\nu(z)^l b_1 + \ldots + \sum_{l=0}^{i_m} \gamma_m(z^{i_m-l}f)\nu(z)^l b_m ,$$

$0 \leq k_1, i_1, \ldots, i_m \leq k-1,$

where $d_1, \ldots, d_{k+k_1}$ is a higher point derivation at 0 and the linear functionals θ_j defined by $\theta_j(f) = \gamma_j(z^{i_j}f)$, $j = 1, \ldots, m$, are discontinuous point derivations at 0.

PROOF: Since $z^k \in M_{n,k-1} = \overline{\ker\nu} = \nu^{-1}(\mathfrak{I}(\nu))$ and $\mathfrak{I}(\nu)^2 = \{0\}$, the multiplication operator $\nu(z): \mathfrak{I}(\nu) \to \mathfrak{I}(\nu)$ is nilpotent of index less than or equal to k. So we may choose a basis B for $\mathfrak{I}(\nu)$ of the form

$$B = \{\nu(z)^k, \ldots, \nu(z)^{k+k_1}, b_1, \nu(z)b_1, \ldots, \nu(z)^{i_1}b_1, \ldots, b_m, \nu(z)b_m, \ldots, \nu(z)^{i_m}b_m\}$$

where $0 \leq k_1, i_1, \ldots, i_m \leq k-1$. Let $f \in C^n[0,1]$. Consider the Taylor expansion

$$f = \sum_{i=0}^{k-1} \delta_i(f)z^i + Rf,$$

where $Rf \in M_{n,\,k-1} = \overline{\ker\nu}$. Since $\nu(Rf) \in \mathfrak{I}(\nu)$ we can write

$$\nu(f) = \sum_{l=0}^{k-1} \delta_l(f)\nu(z)^l + \sum_{l=k}^{k+k_1} d_l(f)\nu(z)^l + \sum_{l=0}^{i_1} \gamma_{1,l+1}(f)\nu(z)^l b_1 + \ldots +$$

$$+ \sum_{l=0}^{i_m} \gamma_{m,l+1}(f)\nu(z)^l b_m$$

We make the following observations:

(i) The coefficient functionals $d_k, \ldots, d_{k+k_1}, \gamma_{1,1}, \ldots, \gamma_{1,i_1+1}, \ldots, \gamma_{m,1}, \ldots, \gamma_{m,i_m+1}$ are discontinuous. To see this, consider $\gamma_{1,1}$. Since $b_1 \in \mathfrak{I}(\nu)$, there exist $f_j \Rightarrow 0$ in $C^n[0,1]$ with $\nu(f_j) \Rightarrow b_1$. We have

$$\nu(f_j) - b_1 = \sum_{l=0}^{k-1} \delta_l(f_j)\nu(z)^l + \sum_{l=k}^{k+k_1} d_l(f_j)\nu(z)^l + (\gamma_{1,1}(f_j) - 1)b_1 +$$

$$+ \sum_{l=1}^{i_1} \gamma_{1,l+1}(f_j)\nu(z)^l b_1 + \sum_{l=0}^{i_2} \gamma_{2,l+1}(f_j)\nu(z)^l b_2 + \ldots + \sum_{l=0}^{i_m} \gamma_{m,l+1}(f_j)\nu(z)^l b_m$$

Since $\nu(f_j)-b_1 \Rightarrow 0$ we must have all coefficients tending to zero as $j\to\infty$, in particular $\lim_{j\to\infty}\gamma_{1,1}(f_j)=1$, which implies that $\gamma_{1,1}$ is discontinuous. The same argument works for the other coefficient functionals $d_k,\dots, d_{k+k_1}, \gamma_{1,2},\dots,\gamma_{1,i_1+1},\dots,\gamma_{m,1},\dots,\gamma_{m,i_m+1}$.

(ii) Since $z^k M_{n,k-1} = M^2_{n,k-1} \subseteq \ker\nu$ (by 3.2) all the above functionals vanish on $z^k M_{n,k-1}$. For a notational purpose we set $d_l=\delta_l$ for $l=0, 1,\dots, k-1$. Let $f, g \in C^n[0,1]$. Using the fact that $\nu(z)^{k+k_1+1}=0$ and $\mathfrak{I}(\nu)^2=\{0\}$ (by 1.4), we have

$$\nu(f)\nu(g)=\sum_{l=0}^{k+k_1}\left[\sum_{j=0}^{l} d_j(f)\,d_{l-j}(g)\right]\nu(z)^l$$
$$+\sum_{l=0}^{i_1}\left[\sum_{j=0}^{l} d_j(f)\gamma_{1,l+1-j}(g)+\gamma_{1,l+1-j}(f)\,d_j(g)\right]\nu(z)^l b_1+\dots$$
$$+\sum_{l=0}^{i_m}\left[\sum_{j=0}^{l} d_j(f)\gamma_{m,l+1-j}(g)+\gamma_{m,l+1-j}(f)\,d_j(g)\right]\nu(z)^l b_m$$

Since $\nu(f)\nu(g)=\nu(fg)$ we have

(iii) $d_l(fg)=\sum_{j=0}^{l} d_j(f)\,d_{l-j}(g)$ for $0\le l\le k+k_1+1$, so $d_1,\dots, d_{k+k_1}$ is a higher point derivation at 0.

(iv) For $j=1,\dots, m$ and $l=1,\dots, i_j$ we have $\gamma_{j,l+1}(z^i)=0$, $i=0, 1, 2,\dots$. Because $\gamma_{j,l+1}(z^i)=0$, for $i=0, 1, \dots, k+k_1$ since $d_j(z^i)=0$ if $i\neq j$, $d_j(z^i)=1$ if $i=j$, and $\operatorname{span}\{e, \nu(z),\dots, \nu(z)^{k-1}\}\oplus \operatorname{span}B=\mathfrak{B}$. $\gamma_{j,l+1}(z^i)=0$, for $i\ge k+k_1+1$ since $\nu(z)^{k+k_1+1}=0$.

Combining (ii) and (iv) we see that all the $\gamma_{j,l+1}$ vanish on $z^kC^n[0,1]$.

(v) For $j=1,\dots, m$ and $l=1,\dots, i_j$ we have

$$\gamma_{j,l+1}(fg)=\sum_{s=0}^{l}\delta_s(f)\gamma_{j,l+1-s}(g)+\gamma_{j,l+1-s}(f)\delta_s(g)$$

so that

$$\gamma_{j,i_j+1}(z^{i_j-l}f)=\gamma_{j,l+1}(f),\ f\in C^n[0,1],\ j=1,\dots, m\ .$$

We take $\gamma_j=\gamma_{j,i_j+1}$, $j=1,\dots, m$. Letting $l=0$ in (v), we note that the linear functionals θ_j, $(j=1,\dots, m)$, defined by $\theta_j(f)=\gamma_{j,1}(f)=\gamma_j(z^{i_j}f)$ are discontinuous point derivations at 0.

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