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INVARIANTS OF NUMBER FIELDS RELATED TO CENTRAL EMBEDDING PROBLEMS

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ABSTRACT. Every central embedding problem over a number field becomes solvable after enlarging its kernel in a certain way. We show that these enlargements can be arranged in a universal way.

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1. CENTRAL EMBEDDING PROBLEMS.

Let K be a number field and let p be a prime number. Then there is a smallest natural number $t = t(k,p)$ depending only on k and p, the so called p-exponent of k, with the following properties:

(1) Every central embedding problem $E_m = E(G,Z/p^m,c)$ for the absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$ of k, where $G = \mathrm{Gal}(K/k)$ is the Galois group of a finite Galois subextension K/k of $\bar{k}/k$ which is ramified only at p and ∞ and where $k(\mu_{p^m})/k$ is cyclic, has exponent $<2m + t$. Recall that E_m is solvable, i.e. there is an epimorphism $\Psi : G_k \to G(c)$ of G_k onto the central group extension G(c) defined by the co-cycle $c:G \times G \to Z/p^m$ such that Ψ composed with the natural map $G(c) \to G$ yields the given epimorphism $G_k \to G$, if and only if the class of (c) becomes trivial in the Brauer group $\mathrm{Br}(k(\mu_{p^m}))$ of $k(\mu_{p^m})$, μ_{p^m} = group of roots of unity of $\bar{k}^*$ of order dividing p^m; this means that if $\chi_m:Z/p^m \to \mu_{p^m}$ is an isomorphism then (c) becomes trivial under the map

$$\tilde{\chi}_m:H^2(G,Z/p^m) \xrightarrow{\inf} H^2(G_k,Z/p^m) \xrightarrow{\mathrm{res}} H^2(G_k(\mu_{p^m})Z/p^m) \xrightarrow{\chi_m^*} H^2(G_k(\mu_{p^m}),\bar{k}^*) \cong \mathrm{Br}(k(\mu_{p^m}))$$

where χ^*_m is the map induced by χ_m on cohomology see Hoechsmann, [1]). The exponent of E_m is the smallest natural number $n > m$ such that the embedding problem E_n which is obtained from E_m by considering the co-cycle $c:G \times G \to Z/p^m \to Z/p^n$ is solvable.

In order to prove (1), choose for any natural number $\hat{m} > m$ an isomorphism $\chi_{\hat{m}}: Z/p^{\hat{m}} \to \mu_{p^{\hat{m}}}$ such that $\chi_{\hat{m}}^{p^{\hat{m}-m}} = \chi_m$. Then we have a map $\hat{\chi}_{\hat{m}}: H^2(G,Z/p^{\hat{m}}) \to Br(k(\mu_{p^{\hat{m}}}))$, and the resulting diagram relating $\hat{\chi}_m$ and $\hat{\chi}_{\hat{m}}$ commutes. Since $\hat{\chi}_{\hat{m}}((c))$ can be represented by a Galois co-cycle all of whose values are roots of unity, the algebra class $\hat{\chi}_{\hat{m}}((c))$ splits and only if it splits locally at all places above p and ∞ and this is the case if it splits at ∞ and

$$(k_v(\mu_{p^{\hat{m}}}):k_v(\mu_{p^m})) \equiv 0 \bmod p^m \text{ for all } v \text{ above } p;$$

(see classels [2], p. 191, 10.5 ff). It is clearly possible to find a smallest integer $d = d(k,p)$ depending only on k and p such that $\hat{\chi}_{\hat{m}}((c))$ splits with $\hat{m} = 2m + d$. For instance, for $k = Q$ we can take $d = d(Q,p) = 0$ for all p. The p-exponent of E_m is the smallest natural number $n > m$ such that the induced embedding problem E_n has a solution which is ramified only at p and ∞. The smallest integer $s > 0$ such that the p-exponent of every E_m is $\leq 2m + s$ is called the strong p-exponent of k (if it exists).

(2) If p does not divide the class number of $Q(\mu_p)$ then the strong p-exponent of every cyclotomic field $k = Q(\mu_{p^l})$ exists and is equal to its (usual) p-exponent.

This can be shown as follows: Let E_m be a central embedding problem for G_k. Then for $t = t(k,p)$ the induced embedding problem E_{2m+t} is solvable. The assumption implies that p does not divide the class number of $Q(\mu_{p^l})$ for every l (see Iwasawa [3]). Therefore the Galois theoretic obstruction to the existence of a solution which is unramified outside p and ∞ as described in Neukirch [4], (8.1), is trivial.

The p-adic Leopoldt conjecture for k implies that $H^2(G_k(p),Q/Z) = 0$, where $G_k(p)$ is the Galois group of the maximal p-extension k^p/k which is unramified outside p and ∞. This shows that every central embedding problem E_m for $G_k(p)$ has finite p-exponent, (see Opolka [5], (5.2)). Does this imply that the strong p-exponent of k is finite? If so, how is it related to the usual p-exponent of k? Conversely, if the strong p-exponent of k is finite then $H^2(G_k(p),Q/Z) = 0$ and the p-adic Leopoldt conjecture holds for k.

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