

INVARIANTS OF NUMBER FIELDS RELATED TO CENTRAL EMBEDDING PROBLEMS

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ABSTRACT. Every central embedding problem over a number field becomes solvable after enlarging its kernel in a certain way. We show that these enlargements can be arranged in a universal way.

KEY WORDS AND PHRASES. Central embedding problems, strict cohomological dimension, Leopoldt-conjecture.

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1. CENTRAL EMBEDDING PROBLEMS.

Let K be a number field and let p be a prime number. Then there is a smallest natural number $t = t(k, p)$ depending only on k and p , the so called p -exponent of k , with the following properties:

(1) Every central embedding problem $E_m = E(G, Z/p^m, c)$ for the absolute Galois group $G_k = \text{Gal}(\bar{k}/k)$ of k , where $G = \text{Gal}(K/k)$ is the Galois group of a finite Galois subextension K/k of \bar{k}/k which is ramified only at p and ∞ and where $k(\mu_m)/k$ is cyclic, has exponent $\leq 2m + t$. Recall that E_m is solvable, i.e. there is an epimorphism $\Psi: G_k \rightarrow G(c)$ of G_k onto the central group extension $G(c)$ defined by the co-cycle $c: G \times G \rightarrow Z/p^m$ such that Ψ composed with the natural map $G(c) \rightarrow G$ yields the given epimorphism $G_k \rightarrow G$, if and only if the class of (c) becomes trivial in the Brauer group $\text{Br}(k(\mu_m))$ of $k(\mu_m)$, $\mu_m =$ group of roots of unity of \bar{k}^* of order dividing p^m ; this means that if $\chi_m: Z/p^m \rightarrow \mu_m$ is an isomorphism then (c) becomes trivial under the map

$$\tilde{\chi}_m: H^2(G, Z/p^m) \xrightarrow{\text{inf}} H^2(G_k, Z/p^m) \xrightarrow{\text{res}} H^2(G_k(\mu_m), Z/p^m) \xrightarrow{\chi_m^*} H^2(G_k(\mu_m), \bar{k}^*) \cong \text{Br}(k(\mu_m))$$

where χ_m^* is the map induced by χ_m on cohomology see Hoechsmann, [1]). The exponent of E_m is the smallest natural number $n > m$ such that the embedding problem E_n which is obtained from E_m by considering the co-cycle $c: G \times G \rightarrow Z/p^m \rightarrow Z/p^n$ is solvable.

In order to prove (1), choose for any natural number $\hat{m} > m$ an isomorphism $\chi_{\hat{m}}: \mathbb{Z}/p^{\hat{m}} \rightarrow \mu_{p^{\hat{m}}}$ such that $\chi_{\hat{m}}^{p^{\hat{m}-m}} = \chi_m$. Then we have a map $\hat{\chi}_{\hat{m}}: H^2(G, \mathbb{Z}/p^{\hat{m}}) \rightarrow \text{Br}(k(\mu_{p^{\hat{m}}}))$, and the resulting diagram relating $\hat{\chi}_m$ and $\hat{\chi}_{\hat{m}}$ commutes. Since $\hat{\chi}_{\hat{m}}((c))$ can be represented by a Galois co-cycle all of whose values are roots of unity, the algebra class $\hat{\chi}_{\hat{m}}((c))$ splits and only if it splits locally at all places above p and ∞ and this is the case if it splits at ∞ and

$$(k_v(\mu_{p^{\hat{m}}}) : k_v(\mu_{p^m})) \equiv 0 \pmod{p^m} \text{ for all } v \text{ above } p;$$

(see classfields [2], p. 191, 10.5 ff). It is clearly possible to find a smallest integer $d = d(k, p)$ depending only on k and p such that $\hat{\chi}_{\hat{m}}((c))$ splits with $\hat{m} = 2m + d$. For instance, for $k = \mathbb{Q}$ we can take $d = d(\mathbb{Q}, p) = 0$ for all p . The p -exponent of E_m is the smallest natural number $n > m$ such that the induced embedding problem E_n has a solution which is ramified only at p and ∞ . The smallest integer $s > 0$ such that the p -exponent of every E_m is $< 2m + s$ is called the strong p -exponent of k (if it exists).

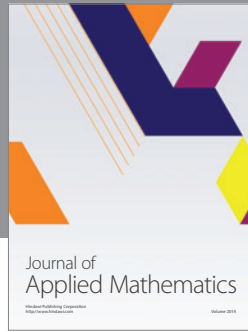
(2) If p does not divide the class number of $\mathbb{Q}(\mu_p)$ then the strong p -exponent of every cyclotomic field $k = \mathbb{Q}(\mu_{p^l})$ exists and is equal to its (usual) p -exponent. This can be shown as follows: Let E_m be a central embedding problem for G_k . Then for $t = t(k, p)$ the induced embedding problem E_{2m+t} is solvable. The assumption implies that p does not divide the class number of $\mathbb{Q}(\mu_{p^l})$ for every l (see Iwasawa [3]).

Therefore the Galois theoretic obstruction to the existence of a solution which is unramified outside p and ∞ as described in Neukirch [4], (8.1), is trivial.

The p -adic Leopoldt conjecture for k implies that $H^2(G_k(p), \mathbb{Q}/\mathbb{Z}) = 0$, where $G_k(p)$ is the Galois group of the maximal p -extension k^p/k which is unramified outside p and ∞ . This shows that every central embedding problem E_m for $G_k(p)$ has finite p -exponent, (see Opolka [5], (5.2)). Does this imply that the strong p -exponent of k is finite? If so, how is it related to the usual p -exponent of k ? Conversely, if the strong p -exponent of k is finite then $H^2(G_k(p), \mathbb{Q}/\mathbb{Z}) = 0$ and the p -adic Leopoldt conjecture holds for k .

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