Internat. J. Math. & Math. Sci.
VOL. 14 NO. 2 (1991) 411-412

ON A FIXED POINT THEOREM OF PATHAK

BRIAN FISHER
Department of Mathematics
The University
Leicester LE1 7RH
England

(Received October 11, 1989)

ABSTRACT. It is shown that the continuity of the mapping in Pathak's fixed point theorem for normed spaces is not necessary.

KEY WORDS AND PHRASES. Normed space, Fixed Point.
1980 AMS SUBJECT CLASSIFICATION CODE. 47H10.

1. INTRODUCTION AND MAIN RESULTS.

In [1] Pathak gives the following definitions:

DEFINITION 1. Let X be a normed vector space; then T, a self mapping of X is called a 'generalized contractive mapping' if

$$\|Tx-Ty\| \le q \max\Big\{\|x-y\|, \frac{\|x-Tx\|[1-\|x-Ty\|]}{1+\|x-Tx\|}, \frac{\|x-Ty\|[1-\|x-Tx\|]}{1+\|x-Ty\|}, \frac{\|y-Tx\|[1-\|y-Ty\|}{1+\|Tx-y\|}, \frac{\|y-Ty\|[1-\|Tx-y\|]}{1+\|y-Ty\|}, \tag{1.1}$$

for all x,y in X, where $0<q<1$.

DEFINITION 2. Let T be a self mapping of a Banach space X. The Mann iterative process associated with T is defined in the following manner:

Let x_0 be in X and set $x_{n+1}=(1-c_n)\,x_n+c_nTx_n$, for $n\ge 0$, where c_n satisfies (i) $c_o=1$, (ii) $0<c_n<1$ for $n>0$, (iii) $\lim_{n\to\infty} c_n=h>0$.

He then proves the following theorem:

THEOREM. Let X be a closed, convex subset of a normed space N, let T be a generalized contractive mapping of X with T continuous on X, and let $\{x_n\}$, the sequence of Mann iterates associated with T, be the same as defined above where $\{c_n\}$ satisfies (i), (ii) and (iii). If $\{x_n\}$ converges in X, then it converges to a fixed point of T.

Pathak finally asks if the continuity of T is necessary in the theorem for T to have a fixed point.

The answer is in the affirmative. To see this, note that if T is a generalized contractive mapping then T also satisfies the inequality

$$\|Tx - Ty\| \leq q \max\{\|x - y\|, \|x - Tx\|, \|x - Ty\|, \|y - Tx\|, \|y - Ty\|\} \tag{1.2}$$

for all x, y in X, where $0 < q < 1$.

Using inequality (1.2) now instead of inequality (1.1) to simplify the work, it follows in exactly the same way as in Pathak's proof of the theorem that if $\lim_{n \to \infty} x_n = z$, then

$$\|z - Tz\| \leq \|z - x_{n+1}\| + (1 - c_n)\|x_n - Tz\|$$
$$+ c_n q \max\{\|x_n - z\|, \|x_n - Tx_n\|, \|x_n - Tz\|, \|Tx_n - z\|, \|z - Tz\|\} \tag{1.3}$$

It now follows from the definition of x_n in Definition 2 that

$$Tx_n = \frac{x_{n+1} - (1 - c_n)x_n}{c_n}$$

and so

$$\lim_{n \to \infty} Tx_n = z.$$

On letting n tend to infinity in inequality (1.3) we now have

$$\|z - Tz\| \leq (1 - h)\|z - Tz\| + hq \max\{0, \|z - Tz\|\}$$

$$= (1 - h + hq)\|z - Tz\|,$$

where $1 - h + hq < 1$. Thus $Tz = z$.

REFERENCES

1. PATHAK, H.K., Some Fixed Point Theorems on Contractive Mappings, Bull. Cal. Math. Soc. 80, 183.