Internat. J. Math. & Math. Sci.
VOL. 14 NO. 4 (1991) 675-678

ON THE THREE-DIMENSIONAL CR-SUBMANIFOLDS OF THE SIX-DIMENSIONAL SPHERE

M.A. BASHIR

Mathematics Department
College of Science
King Saud University
P.O. Box 2455
Riyadh 11451 Saudi Arabia

(Received December 13, 1989 and in revised form March 27, 1990)

ABSTRACT. We show that the six-dimensional sphere does not admit three-dimensional totally umbilical proper CR-submanifolds.

KEY WORDS AND PHRASES. Totally umbilical submanifolds, totally real submanifolds, CR-submanifolds, almost complex structure.
1980 AMS SUBJECT CLASSIFICATION CODE. 53C40; 53C55.

1. INTRODUCTION. The six-dimensional unit sphere $S^6(1)$ has a nearly Kaehler structure J constructed in a natural way by making use of Cayley division algebra [3]. It is because of this nearly Kaehler, non-Kaehler structure, that $S^6(1)$ has drawn the attention. In particular, almost complex submanifolds, CR-submanifolds and totally real submanifolds of $S^6(1)$ have been considered by A. Gray [4], K. Sekigawa and N. Ejiri [2]. For three-dimensional totally real submanifolds of $S^6(1)$ of constant curvature, N. Ejiri proved the following [2].

THEOREM 1. Let M be a 3-dimensional totally real submanifold of constant curvature c in $S^6(1)$. Then $c = 1$ (totally geodesic) or $c = \frac{1}{16}$ (minimal).

In this paper we consider 3-dimensional CR-submanifolds of $S^6(1)$. We prove the following result:

THEOREM 2. There are no 3-dimensional totally umbilical proper CR-submanifolds in $S^6(1)$.

2. PRELIMINARIES.

Let C_+ be the set of all purely imaginary Cayley numbers. The C_+ can be viewed as a 7-dimensional linear subspace $\mathbb{R}^7$ of $\mathbb{R}^8$. Consider the unit hypersphere which is centered at the origin

$$S^6(1) = \{x \varepsilon C_+ \mid \; < x, x > \; = 1\}.$$

The tangent space $T_x S^6$ of $S^6(1)$ at a point x may be identified with the affine subspace of C_+ which is orthogonal to x. On $S^6(1)$ define a (1,1)-tensor field J by putting

$$J_x U = x \times U,$$

where the above product is defined as in [3] for $x \varepsilon S^6(1)$ and $U \varepsilon T_x S^6$.

The above tensor field J determines an almost complex structure (i.e., $J^2 = -Id$) on $S^6(1)$. The compact simple lie group of automorphisms G_2 acts transitively on $S^6(1)$ and preserves both J and the standard metric on $S^6(1)$, [3].

Now let G be the (2,1)-tensor field on $S^6(1)$ defined by

$$G(X,Y) = (\bar{\nabla}_X J)Y$$

where $\bar{\nabla}$ is the Levi-Civita connection on $S^6(1)$ and $X, Y \varepsilon T_x S^6$.

Since $\bar{\nabla}_X J$ is skew-symmetric with respect to the Hermitian metric g on $S^6(1)$, it follows that G has the following property

$$g(G(X,Y),Z) + g(G(X,Z),Y) = 0 \tag{2.1}$$

where $X, Y, Z \varepsilon \mathfrak{X}(S^6)$.

A submanifold M of of $\dim(2p+q)$ in $S^6(1)$ is called a CR-submanifold if there exists a pair of orthogonal complementary distributions D and $\overset{\perp}{D}$ such that $JD = D$ and $J\overset{\perp}{D} \subset \nu$, where ν is the normal bundle of M and $\dim D = 2p$, $\dim \overset{\perp}{D} = q$[1]. Thus the normal bundle ν splits as $\nu = J\overset{\perp}{D} \oplus \mu$, where μ is invariant sub-bundle of ν under J.

A CR-submanifold is said to be proper if neither $D = \{0\}$ nor $\overset{\perp}{D} = \{0\}$.

We denote by $\nabla, \bar{\nabla}, \overset{\perp}{\nabla}$ the Riemannian connections on M, S^6 and the normal bundle respectively. They are related by Gauss formula and Weingarten formula:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.2}$$

$$\bar{\nabla}_X N = -A_N X + \overset{\perp}{\nabla}_X N \qquad N \varepsilon \nu \tag{2.3}$$

where $h(X,Y)$ and $A_N X$ are the second fundamental forms which are related by

$$g(h(X,Y),N) = g(A_N X, Y) \tag{2.4}$$

X and Y are vector fields on M.

Now a CR-submanifold is said to be totally umbilical if $h(X,Y) = g(X,Y)H$ where $H = \frac{1}{n}$ (trace h) is the mean curvature vector. If M is a totally umbilical CR-submanifold, then equations (2) and (3) become

$$\bar{\nabla}_X Y = \nabla_X Y + g(X,Y)H \tag{2.5}$$

$$\bar{\nabla}_X N = -g(H,N)X + \overset{\perp}{\nabla}_X N \tag{2.6}$$

Let R be the curvature tensor associated with ∇. Then the equation of Gauss is given by

$$R(X,Y;Z,W) = g(X,Z)g(Y,W) - g(Y,Z)g(X,W)$$

$$+ g(h(X,Z),h(Y,W)) - g(h(Y,Z),h(X,W))$$

It is known that for X,Y in D, $G(X,Y) = 0$, and $G(W,W) = 0$ for all $W \varepsilon \mathfrak{X}(S^6)$.

3. <u>3-DIMENSIONAL CR-SUBMANIFOLDS OF $S^6(1)$:</u>

Let M be a 3-dimensional totally umbilical proper CR-submanifold of $S^6(1)$. Since M is proper, $D \neq \{0\}$ and $\overset{\perp}{D} \neq \{0\}$. Then since $\dim M = 3$, we have $\dim D = 2$ and $\dim \overset{\perp}{D} = 1$.

We have the following:

LEMMA 1. If M is a 3-dimensional totally umbilical proper CR-submanifold of $S^6(1)$, then $H \varepsilon J\overset{\perp}{D}$.

PROOF. For $X, Y \neq 0$ in D we use equation (2.5) and the equation $J\bar{\nabla}_X Y = \bar{\nabla}_X JY$ to get

$$J\nabla_X Y + g(X,Y)JH = \nabla_X JY + g(X,JY)H. \tag{3.1}$$

Taking inner product in (3.1) with $N\varepsilon\mu$ we have

$$g(X,Y)g(JH,N) = g(X,JY)g(H,N) \tag{3.2}$$

In particular, if we let $Y = JX$ in (3.2) we get

$$\| X \| g(H,N) = 0$$

From which it follows that $H\varepsilon J\overset{\perp}{D}$.

LEMMA 2. If M is a 3-dimensional totally umbilical CR-submanifold of $S^6(1)$, the $\| H \|$ is constant.

PROOF. Using (2.7) and the equation $h(X,Y) = g(X,Y)H$ we get

$$R(X,Y;Z,W) = (1 + \| H \|^2)\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\} \tag{3.3}$$

Then since $\dim M = 3$, we invoke Schur's theorem to conclude that $(1 + \| H \|^2)$ is constant. Thus $\| H \|$ is constant.

4. PROOF OF THEOREM 2.

In this section let $\{X, JX, Z\}$ denote an orthonormal frame field for the 3-dimensional totally umbilical CR-submanifold M of $S^6(1)$. The unit vector fields X, JX are in D and the unit vector field Z is in $\overset{\perp}{D}$. Since M is totally umbilical, the equation $h(X,Y) = g(X,Y)H$ implies that

$$h(X,JX) = h(X,Z) = h(JX,Z) = 0$$

and

$$h(X,X) = h(JX,JX) = h(Z,Z) = H \tag{4.1}$$

We know from the previous Lemma that $H\varepsilon J\overset{\perp}{D}$. Since $\dim J\overset{\perp}{D} = 1$, then one can write $H = \alpha JZ$ for some smooth function α on M. Therefore

$$h(X,X) = h(JX,JX) = h(Z,Z) = \alpha JZ$$

Using equation (2.4) with $N = JZ$ we get

$$A_{JZ}X = \alpha X, \qquad A_{JZ}JX = \alpha JX, \qquad A_{JZ}Z = \alpha Z \tag{4.2}$$

So the frame field $\{X, JX, Z\}$ diagonalizes A. Now in $S^6(1)$ we have equation (2.1) i.e. $g((\bar\nabla_X J)Y, Z) + g((\bar\nabla_X J)Z, Y) = 0$ for any $X, Y, Z\varepsilon\mathfrak{X}(S^6)$. Since for $X, Y\varepsilon D$ $(\bar\nabla_X J)Y = 0$, then using this equation with $Y = JX$ for our orthonormal frame field $\{X, JX, Z\}$ in M, we get

$$g((\bar\nabla_X J)Z, JX) = 0 \tag{4.3}$$

Using equation (2.5), (4.3) and (2.6) with the fact that $H\varepsilon J\overset{\perp}{D}$ and $(\bar\nabla_X J)Z = \bar\nabla_X JZ - J\bar\nabla_X Z$ we get

$$g(\nabla_X Z, X) = 0 \tag{4.4}$$

Again using equation (2.5) and (2.6) in equation (2.1) with $Y = X$, we get

$$g(\nabla_X Z, JX) = \alpha \tag{4.5}$$

Also using equation (2.1) and $(\bar{\nabla}_{JX}J)Z = \bar{\nabla}_{JX}JZ - J\bar{\nabla}_{JX}Z$ we get

$$g(\nabla_{JX}Z, X) = -\alpha \tag{4.6}$$

Switching the role of X and Y in equation (2.1) and letting $Y = JX$ we obtain

$$g(\nabla_{JX}Z, JX) = 0 \tag{4.7}$$

Now using the equation $g((\bar{\nabla}_X J)X, JZ) = 0$ and $g(\bar{\nabla}_{JX}J)x, z) = 0$ we get

$$g(\nabla_X X, Z) = 0, \qquad g(\nabla_{JX}JX, Z) = 0 \tag{4.8}$$

From the equation $(\bar{\nabla}_Z J)Z = 0$, using equation (4.1) and (4.2) and the fact that $\nabla_Z Z \varepsilon D$, we get

$$\nabla_Z Z = 0, \qquad \overset{\perp}{\nabla}_Z JZ = 0 \tag{4.9}$$

Using equations (4.4), (4.5), (4.6), (4.7), (4.8) and the first part of equation (4.9) we can write the local equations for the frame field $\{X, JX, Z\}$ as follows:

$$\nabla_X Z = \alpha JX, \qquad \nabla_{JX}Z = -\alpha X, \qquad \nabla_Z Z = 0$$

$$\nabla_X X = aJX, \qquad \nabla_{JX}X = -bJX + \alpha Z, \qquad \nabla_Z X = cJX$$

$$\nabla_X JX = -aX - \alpha Z, \qquad \nabla_{JX}JX = bX, \qquad \nabla_Z JX = -cX \tag{4.10}$$

for some smooth functions a, b and c.

The curvature tensor R is given by

$$R(X,Y;Z,W) = < \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W >$$

Then using this equation with the help of equations (4.10) we get $R(X,Z,Z,X) = \alpha^2$, $\alpha = \| H \|$. But from equation (3.3) we know that $R(X,Z,Z,X) = -(1+\alpha^2)$. This is a contradiction and hence $S^6(1)$ cannot admit a 3-dimensional totally umbilical proper CR-submanifolds.

REFERENCES

1. BEJANCU, A., CR-submanifolds of a Kaehler manifold, Proc. Amer. Math. Soc. 69(1978), 135-142.
2. EJIRI, N., Totally real submanifolds in a 6-sphere, Proc. Amer. Math. Soc. 83(1981), 759-763.
3. FUKAMI, T. and ISHIHARA, S., Almost Hermitian structure on S^6, Tohoku Math. J. 7(1955), 151-156.
4. GRAY, A., Almost complex submanifolds of six sphere, Proc. Amer. Math. Soc. 20(1969), 277-279.
5. SEKIGAWA, K., Almost complex submanifolds of a 6-dimensional sphere, Kodai Math. J., 6(1983), 174-185.