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ON LINDELÖF LATTICES AND SEPARATION

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ABSTRACT. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. Some general properties of Lindelöf, regular as well as normal lattices are investigated for their measure implications and their relationship to separation properties. Moreover, we show that the generalized Wallman replete space and the generalized Wallman prime complete space are Lindelöf spaces if and only if certain measure relationships hold on $\mathcal{L}$.

KEY WORDS AND PHRASES. Lindelöf lattice, 0-1 valued measures, disjunctive lattice, countably compact, normal lattice, prime complete, premeasure, delta lattice, replete, regular, slightly normal, I-lattice.
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1. INTRODUCTION. In this paper we consider in detail, Lindelöf lattices and their relationship to separation properties, and also to certain measure properties. In particular, we show that the generalized Wallman replete space is a Lindelöf space if and only if a certain measure relationship holds on $\mathcal{L}$, and, similarly, the generalized Wallman prime complete space is Lindelöf if and only if another measure relationship holds on $\mathcal{L}$.

Before doing this, we investigate some general properties of regular, Lindelöf lattices as well as normal and slightly normal lattices. We are mainly concerned with measure implications of these properties, and also lattice separation properties that may result.

We adhere to standard lattice and measure terminology and notation (See e.g. [1], [2], [3], and [7]), and we give some relevant background material in Section 2 for the reader's convenience.

2. BACKGROUND AND NOTATION. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. It is assumed that $\emptyset$ and $X \in \mathcal{L}$, although this is not necessary for some of our results.

MEASURE TERMINOLOGY. We now introduce some measure terminology and results.

1. $a(\mathcal{L})$ is the algebra generated by $\mathcal{L}$.

2. $I(\mathcal{L})$ is the set of all 0-1 valued finitely additive measures defined on a $(\mathcal{L})$. Also, we note that there exists a 1-1 correspondence between $\mu \in I(\mathcal{L})$ and prime $\mathcal{L}$-filters given by $F = \{ \in \mathcal{L};\ \mu(L) = 1 \}$.

3. $I_R(\mathcal{L})$ is the set of all $\mu \in I(\mathcal{L})$ which are $\mathcal{L}$-regular; that is, for $A \in a(\mathcal{L})$, $\mu(A) = \sup \mu(L)$ where $L \subset A$, $L \in \mathcal{L}$. Observe that the correspondence given in 2 gives a 1-1 correspondence between $I_R(\mathcal{L})$ and $\mathcal{L}$-ultrafilters.

4. $I_\sigma(\mathcal{L})$ is the set of all 0-1 σ-smooth finitely additive measures on $\mathcal{L}$. $\mu \in I(\mathcal{L})$ is σ-smooth if for $L_n \in \mathcal{L}$, $L_n \downarrow \emptyset$ then $\lim_{n\to\infty} \mu(L_n) = 0$. Also, we note that $\mu \in I_\sigma(\mathcal{L})$ if and only if the associated prime $\mathcal{L}$-filter given in 2 has the countable intersection property (c.i.p.).

5. $I^\sigma(\mathcal{L})$ is the set of all 0-1 σ-smooth finitely additive measures on a $(\mathcal{L})$. A 0-1 measure μ is σ-smooth on a $(\mathcal{L})$ if for $A_n \in \mathcal{L}$, $A_n \downarrow \emptyset$, then $\lim_{n\to\infty} \mu(A_n) = 0$.

6. It is worthwhile noting that if $\mu \in I_R(\mathcal{L})$ then $\mu \in I^\sigma(\mathcal{L})$ if and only if $\mu \in I_\sigma(\mathcal{L})$.

7. $I^\sigma_R(\mathcal{L})$ is the set of all $\mathcal{L}$-regular measures of $I^\sigma(\mathcal{L})$.

8. $\Pi(\mathcal{L})$ is the set of all premeasures on $\mathcal{L}$; i.e., $\pi \in \Pi(\mathcal{L})$ if $\pi: \mathcal{L} \to \{0,1\}$ and $\pi(\emptyset) = 0$, π is monotone and $\pi(A) = \pi(B) = 1$ for $A, B \in \mathcal{L}$ then $\pi(A \cap B) = 1$. Also, it is easy to see that there exists a 1-1 correspondence between $\mathcal{L}$ filters and premeasures on $\mathcal{L}$. In addition, if $\pi \in \Pi(\mathcal{L})$ and $L_n \downarrow \emptyset$, $L_n \in \mathcal{L}$ then $\lim_{n\to\infty} \pi(L_n) = 0$ then we write $\pi \in \Pi_\sigma(\mathcal{L})$.

9. For $\mu \in I(\mathcal{L})$, we define the support of μ to be $S(\mu) = \{\cap L;\ L \in \mathcal{L};\ \mu(L) = 1\}$. Similarly, we define $S(\pi)$ for $\pi \in \Pi(\mathcal{L})$.

10. For $x \in X$, we define $\mu_x(A) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$, $A \in a(\mathcal{L})$ to be the measure concentrated at x.

11. By $\mu \le \nu(\mathcal{L})$, we simply mean $\mu(L) \le \nu(L)$ for all $L \in \mathcal{L}$.

LATTICE TERMINOLOGY. We next present some lattice terminology and results.

1′. $\delta(\mathcal{L})$ is the lattice of all countable intersections of sets from $\mathcal{L}$.

2′. $\tau(\mathcal{L})$ is the lattice of arbitrary intersections of sets from $\mathcal{L}$.

3′. $\mathcal{L}'$ is the complementary lattice of $\mathcal{L}$; i.e., $\mathcal{L}' = \{L';\ L \in \mathcal{L}\}$, where prime denotes complement.

4′. $\mathcal{L}$ is disjunctive if for any $x \in X$ and $L_1 \in \mathcal{L}$; $x \notin L_1$, there exists an $L_2 \in \mathcal{L}$; $x \in L_2'$ and $L_1 \cap L_2 = \emptyset$. Also, we note that $\mathcal{L}$ is disjunctive if and only if $\mu_x \in I_R(\mathcal{L})$ for all $x \in X$.

5′. $\mathcal{L}$ is regular if for any $x \in X$ and $L_1 \in \mathcal{L}$; $x \notin L_1$, there exist $L_2, L_3 \in \mathcal{L}$; $x \in L_2', L_1 \subset L_3'$ and $L_2' \cap L_3' = \emptyset$. Also, $\mathcal{L}$ is regular if and only if $\mu_1 \le \mu_2(\mathcal{L}), \mu_1, \mu_2 \in I(\mathcal{L})$ implies $S(\mu_2) \subset S(\mu_1)$.

6′. $\mathcal{L}$ is normal if for any $L_1, L_2 \in \mathcal{L}; L_1 \cap L_2 = \emptyset$, there exist $L_3, L_4 \in \mathcal{L}; L_1 \subset L_3', L_2 \subset L_4'$ and $L_3' \cap L_4' = \emptyset$. Also, $\mathcal{L}$ is normal if and only if for any $\mu \in I(\mathcal{L})$ there exists a unique $\nu \in I_R(\mathcal{L})$ such that $\mu \le \nu(\mathcal{L})$.

7′. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices of subsets of X. $\mathcal{L}_1$ separates $\mathcal{L}_2$ if $L_1, L_2 \in \mathcal{L}_2; L_1 \cap L_2 = \emptyset$ then, there exist $L_3, L_4 \in \mathcal{L}_1; L_1 \subset L_3, L_2 \subset L_4$ and $L_3 \cap L_4 = \emptyset$.

8′. $\mathcal{L}$ is Lindelöf if $\bigcap_x L_x = \emptyset$ where $L \in \mathcal{L}$ then, for a countable subcollection L_{x_i}, $i = 1, 2, \cdots$ of L, $\bigcap_{i=1}^{\infty} L_{x_i} = \emptyset$. Equivalently, $\mathcal{L}$ is Lindelöf if and only if for each $\pi \in \Pi_\sigma(\mathcal{L}), S(\pi) \ne \emptyset$.

9′. $\mathcal{L}$ is countably compact if every countable covering of X by elements of $\mathcal{L}'$ has a finite subcovering.

10′. $\mathcal{L}$ is replete if for any $\mu \in I^\sigma_R(\mathcal{L})$, $S(\mu) \ne \emptyset$.

11′. $\mathcal{L}$ is a delta lattice (δ-lattice) if $\delta(\mathcal{L}) = \mathcal{L}$.

Further related material can be found in ([4}, [5], [6] and [8]).

3. ON SEPARATION. This section begins with the following observations which follows directly from the definitions.

(1) $\mathcal{L}$ is Lindelöf if and only if $\tau\mathcal{L}$ is Lindelöf; and

(2) if $\mathcal{L}$ is δ and Lindelöf then, $\mathcal{L}$ separates $\tau\mathcal{L}$.

Now we investigate some implications of lattices properties for the correspondent measures.

THEOREM 3.1. If $\mathcal{L}$ is regular and Lindelöf then, $I_\sigma(\mathcal{L}') \subset I_\sigma(\mathcal{L})$.

PROOF. Suppose $I_\sigma(\mathcal{L}') \not\subset I_\sigma(\mathcal{L})$; that is, there exists a $\mu \in I_\sigma(\mathcal{L}')$ but not in $I_\sigma(\mathcal{L})$. Let $\rho(L) = \sup \mu(\hat{L}'), \hat{L}' \subset L, \hat{L}', L \in \mathcal{L}$ which is a premeasure; that is, $\rho \in \Pi(\mathcal{L})$. But since $\mathcal{L}$ is regular then $S(\mu) = S(\rho)$. Also, $\rho \in \Pi_\sigma(\mathcal{L})$ since $\mu \in I_\sigma(\mathcal{L}')$, and since $\mathcal{L}$ is Lindelöf, $S(\rho) \ne \emptyset$ then $S(\mu) \ne \emptyset$.

Let $x \in S(\mu)$, then $\mu \leq \mu_x(\mathcal{L})$ therefore $\mu \in I_\sigma(\mathcal{L})$ this is a contradiction. Thus, $I_\sigma(\mathcal{L}') \subset I_\sigma(\mathcal{L})$.

DEFINITION 3.1. $\mathcal{L}$ is slightly normal if for every $\mu \in I_\sigma(\mathcal{L}')$, there exists a unique $\nu \in I_R(\mathcal{L})$; $\mu \leq \nu(\mathcal{L})$.

THEOREM 3.2. a) If $\delta(\mathcal{L}')$ separates $\mathcal{L}$, then $\mathcal{L}$ is slightly normal; b) Suppose whenever $A \cap B = \emptyset$, $A, B \in \mathcal{L}$, there exist $\widehat{A}, B_n \in \mathcal{L}$ such that $A \subset \widehat{A} \subset \bigcup_{n=1}^{\infty} B_n \subset B'$ then $\delta(\mathcal{L}')$ separates $\mathcal{L}$.

PROOF. Let $\mu \in I_\sigma(\mathcal{L}')$, $\mu \leq \nu_1(\mathcal{L})$, $\mu \leq \nu_2(\mathcal{L})$ where $\nu_1, \nu_2 \in I_R(\mathcal{L})$. If $\nu_1 \neq \nu_2$, there exist $A, B \in \mathcal{L}$, $\nu_1(A) = \nu_2(B) = 1$ and $\nu_2(A) = \nu_1(B) = 0$ then by hypothesis, there exist L'_n, L'_m where $L_n, L_m \in \mathcal{L}$ for all n, m such that $A \subset \cap L'_n$, $B \subset \cap L'_m$ and $(\cap L'_n) \cap (\cap L'_m) = \emptyset$. Since $\nu_1(A) = 1$ and $\mu \leq \nu_1(\mathcal{L})$, then $\mu(L'_n = 1)$ for all n. Similarly, we get $\mu(L'_m) = 1$ for all m. Thus, $\mu(L'_n \cap L'_m = 1$ for all n, m but $\bigcap_{n,m} L'_n \cap L'_m = \emptyset$ which is a contradiction since $\mu \in I_\sigma(\mathcal{L}')$. Thus, $\nu_1 \leq \nu_2$ and so $\mathcal{L}$ is slightly normal; c) the proof is clear.

Next, we consider the following condition, designated as $(*)$

$$\text{Let } A, B \in \mathcal{L};\ A \cap B = \emptyset \text{ then, there exists } A'_i \uparrow \text{ and } B_i \in \mathcal{L};\ A \subset \bigcup_{i=1}^{\infty} A'_i,$$
$$A_i \in \mathcal{L} \text{ and } A'_i \subset B_i \text{ and also } B_i \cap B = \emptyset \text{ for all } i.$$

THEOREM 3.3. If $\mathcal{L}$ is δ and satisfies $(*)$ then, $\mathcal{L}$ is normal.

PROOF. Let $A \cap B = \emptyset$ for $A, B \in \mathcal{L}$ then, there exists $C'_i \uparrow, C_i \in \mathcal{L}; B \subset \bigcup_{i=1}^{\infty} C'_i$ and there exists $D_i \in \mathcal{L}; C'_i \subset D_i$ and $D_i \cap A = \emptyset$ for all i. Let $S_i = A'_i \cap D'_i$ and $H_j = C'_j \cap B'_j$ clearly $S_i, H_j \in \mathcal{L}'$. Then, $S_i \cap H_j = \emptyset$ since $S_i \cap H_j = A'_i \cap D'_i \cap C'_j \cap B'_j$ and $i \leq j$ then clearly $A'_i \subset A'_j \subset B_j$, while if $i > j$ then clearly $C'_j \subset C'_i \subset D_i$. Also, $\bigcup_i S_i = \bigcup_i (A'_i \cap D'_i) \supset A$ since $\bigcup_i A'_i \subset A$ and $D'_i \subset A$ for all f. Similarly, $\bigcup_j H_j \supset B$ and $\bigcup_i S_i$, $\bigcup_j H_j \in \mathcal{L}'$ if $\mathcal{L}$ is δ and so they are disjoint as shown. Thus, $\mathcal{L}$ is normal.

THEOREM 3.4. If $\mathcal{L}$ is δ, regular and Lindelöf then, $\mathcal{L}$ is normal.

PROOF. Let $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$. Since $\mathcal{L}$ is regular, for $x \in A$ there exist $L_x, \widehat{L}_x \in \mathcal{L}$ such that $x \in L'_x \subset \widehat{L}_x$, and $\widehat{L}_x \cap B = \emptyset$. Since $\mathcal{L}$ is Lindelöf there exists a countable number $L_{x_i}, i = 1, 2, \cdots$ of the L_x such that $A \subset \cup L'_x$, and we may assume $L'_{x_i} \uparrow$. Then, $L'_{x_i} \subset \widehat{L}_{x_i}$ and $\widehat{L}_{x_i} \cup B = \emptyset$ so condition $(*)$ is satisfied and so since $\mathcal{L}$ is δ, it follows that $\mathcal{L}$ is normal by Theorem 3.3.

Next, for $\mu \in I(\mathcal{L})$ and $E \subset X$, we define $\mu'(E) = \inf \mu(L')$ where $E \subset L', L \in \mathcal{L}$. And, we say that $\mu \in I_w(\mathcal{L})$ if $\mu(L') = 1, L \in \mathcal{L}$ then $L' \supset \widehat{L} \in \mathcal{L}$ and $u'(\widehat{L}) = 1$. Also, we note that $\mu = \mu'(\mathcal{L})$ if and only if $\mu \in I_R(\mathcal{L})$.

THEOREM 3.5. If $\mathcal{L}$ is normal then, $I_w(\mathcal{L}) = I_R(\mathcal{L})$.

PROOF. i) If $\mu \in I_R(\mathcal{L})$ and $\mu(L') = 1, L \in \mathcal{L}$ then $L' \supset \widehat{L} \in \mathcal{L}, \mu(\widehat{L}) = 1$ but $\mu \leq \mu'(\mathcal{L})$ therefore $\mu'(\widehat{L}) = 1$. Thus, $\mu \in I_w(\mathcal{L})$ and so $I_R(\mathcal{L}) \subset I_w(\mathcal{L})$; ii) In general, we know that $\mu \leq \mu'(\mathcal{L})$. Now, suppose $\mathcal{L}$ is normal and $\mu \in I_w(\mathcal{L})$. Suppose $\mu(L) = 0, L \in \mathcal{L}$ then $\mu(L') = 1$ therefore $L' \supset \widehat{L} \in \mathcal{L}$ and $\mu'(\widehat{L}) = 1$. By normality $L \subset A'$, $\widehat{L} \subset B'$, $A, B \in \mathcal{L}$ and $A' \cap B' = \emptyset$. Since $\mu'(\widehat{L}) = 1$, therefore $\mu(B') = 1$ and hence $\mu'(L) = 0$.

One may easily note that the converse of Theorem 3.5 is not true.

THEOREM 3.6. If $\mu \in I_\sigma(\mathcal{L}') \cap I_w(\mathcal{L})$ and $\delta(\mathcal{L}')$ separates $\mathcal{L}$ then, $\mu \in I_R(\mathcal{L})$.

PROOF. Let $L_1 \in \mathcal{L}$ and $\mu(L_1) = 0$ so $\mu(L'_1) = 1$ then $L'_1 \supset L_2 \in \mathcal{L}$ with $\mu'(L_2) = 1$ since $\mu \in I_w(\mathcal{L})$. Since $\delta(\mathcal{L}')$ separates $\mathcal{L}$, $L_1 \subset \cap A'_n, A_n \in \mathcal{L}, L_2 \subset \cap B'_m$, $B_m \in \mathcal{L}$ and $(\cap A'_n) \cap (\cap B'_m) = \emptyset$ so $\bigcap_{n,m} A'_m \cap B'_m = \emptyset$ (and we may assume $\downarrow$). Therefore, $\mu(A'_n \cap B'_m) = 0$ for n, m big since $\mu \in I_\sigma(\mathcal{L}')$ but $L_2 \subset B'_m$ therefore $1 = \mu'(L_2) \leq \mu'(B'_m) = \mu(B'_m)$ since in general $\mu = \mu'(\mathcal{L}')$

therefore $\mu(A'_n) = 0$ for n big but $L_1 \subset A'_n$ then $L'_1 \supset A_n$ and $\mu(A_n) = 1$. Thus, $\mu \in I_R(\mathcal{L})$.

4. LINDELÖF LATTICES. This section is divided into two parts.

PART A

DEFINITION 4.1. $\mathcal{L}$ is an I-lattice if for every $\pi \in \Pi_\sigma(\mathcal{L})$, there exists a $\mu \in I^\sigma_R(\mathcal{L})$; $\pi \leq \mu(\mathcal{L})$.

THEOREM 4.1. If $\mathcal{L}$ is an I-lattice and replete then, $\mathcal{L}$ is Lindelöf.

PROOF. Let $\pi \in \Pi_\sigma(\mathcal{L})$ then there exists a $\mu \in I^\sigma_R(\mathcal{L})$; $\pi \leq \mu(\mathcal{L})$ since $\mathcal{L}$ is an I-lattice. Since $\mathcal{L}$ is replete $S(\mu) = \cap L \neq \emptyset$ where $\mu(L) = 1$, $L \in \mathcal{L}$. In addition, $S(\pi) \subset S(\mu)$ where $S(\pi) = \cap L_x$ and $\pi(L_x) = 1$, $L_x \in \mathcal{L}$ for all x. Thus, $S(\pi) \neq \emptyset$ and so $\mathcal{L}$ is Lindelöf.

THEOREM 4.2. If $\mathcal{L}$ is countably compact then, $\mathcal{L}$ is an I-lattice.

PROOF. Let $\pi \in \Pi_\sigma(\mathcal{L})$ then there exists a $\mu \in I_R(\mathcal{L})$; $\pi \leq \mu(\mathcal{L})$ but since $\mathcal{L}$ is countably compact $\mu \in I^\sigma_R(\mathcal{L}) = I_R(\mathcal{L})$ and so $\mathcal{L}$ is an I-lattice.

THEOREM 4.3. If $\mathcal{L}$ is disjunctive and Lindelöf then, $\mathcal{L}$ is an I-lattice.

PROOF. Let $\pi \in \Pi_\sigma(\mathcal{L})$. Since $\mathcal{L}$ is Lindelöf then $S(\pi) \neq \emptyset$. Let $x \in S(\pi)$ since $\mathcal{L}$ is disjunctive, $\mu_x \in I_R(\mathcal{L})$ where $\pi \leq \mu_x(\mathcal{L})$ and μ_x is σ-smooth on $\mathcal{L}$ then $\mu_x \in I^\sigma_R(\mathcal{L})$ and so $\mathcal{L}$ is an I-lattice.

Next, we consider $W_\sigma(\mathcal{L}) = \{W_\sigma(L); L \in \mathcal{L}\}$ where $W_\sigma(L) = \{\mu \in I^\sigma_R(\mathcal{L}); \mu(L) = 1\}$. Note that $W_\sigma(\mathcal{L})$ forms a base for the closed sets $\tau W_\sigma(\mathcal{L})$ of $I^\sigma_R(\mathcal{L})$. Also, we may note the following well-known fact:

1) If $\mathcal{L}$ is disjunctive then, $I^\sigma_R(\mathcal{L})$, $W_\sigma(\mathcal{L})$ is replete.

THEOREM 4.4. If $\mathcal{L}$ is disjunctive and an I-lattice then, the topological space $I^\sigma_R(\mathcal{L})$, $\tau W_\sigma(\mathcal{L})$ is Lindelöf.

PROOF. Since $\mathcal{L}$ satisfies conditions for an I-lattice so does $W_\sigma(\mathcal{L})$. Also, since $\mathcal{L}$ is disjunctive then, $W_\sigma(\mathcal{L})$ is replete. Thus, by Theorem 4.1, $W_\sigma(\mathcal{L})$ is Lindelöf and so is $\tau W_\sigma(\mathcal{L})$.

THEOREM 4.5. Assume $\mathcal{L}$ is disjunctive. If $I^\sigma_R(\mathcal{L})$, $\tau W_\sigma(\mathcal{L})$ is Lindelöf, then $\mathcal{L}$ is an I-lattice.

PROOF. Let $\mu \in I^\sigma_R(\mathcal{L})$ and $\mu \notin W_\sigma(L)$, $L \in \mathcal{L}$ then, $\mu(L') = 1$ therefore $L' \supset \widehat{L} \in \mathcal{L}$ and $\mu(\widehat{L}) = 1$ therefore $\mu \in W_\sigma(\widehat{L})$ and $W_\sigma(\widehat{L}) \cap W_\sigma(L) = \emptyset$ therefore $W_\sigma(\mathcal{L})$ is disjunctive. Also, $W_\sigma(\mathcal{L})$ is Lindelöf then, by Theorem 4.3, $W_\sigma(\mathcal{L})$ is an I-lattice. Thus, $\mathcal{L}$ is an I-lattice.

PART B

We now consider the following condition, designated as (**) [For every $\pi \in \Pi_\sigma(\mathcal{L})$, there exists a $\nu \in I_\sigma(\mathcal{L})$; $\pi \leq \nu(\mathcal{L})$] (**).

DEFINITION 4.2. $\mathcal{L}$ is a prime complete if for $\mu \in I_\sigma(\mathcal{L})$, $S(\mu) \neq \emptyset$.

THEOREM 4.6. If $\mathcal{L}$ is prime complete and satisfies (**) then, $\mathcal{L}$ is Lindelöf.

PROOF. If $\mathcal{L}$ satisfies (**) then, for $\pi \in \Pi_\sigma(\mathcal{L})$ there exists a $\nu \in I_\sigma(\mathcal{L})\pi \leq \nu(\mathcal{L})$. By prime completeness of $\mathcal{L}$, $S(\nu) \neq \emptyset$ but $S(\nu) \subset S(\pi)$ then $S(\pi) \neq \emptyset$ and so $\mathcal{L}$ is Lindelöf.

THEOREM 4.7. If $\mathcal{L}$ is countably compact then, (**) is satisfied.

PROOF. Let $\pi \in \Pi_\sigma(\mathcal{L})$ then there exists a $\nu \in I(\mathcal{L}) \subset I_\sigma(\mathcal{L})$ since $\mathcal{L}$ is countably compact; $\pi \leq \nu(\mathcal{L})$ and so (**) is satisfied.

THEOREM 4.8. If $\mathcal{L}$ is Lindelöf then, (**) is satisfied.

PROOF. Let $\pi \in \Pi_\sigma(\mathcal{L})$: $S(\pi) \neq \emptyset$ since $\mathcal{L}$ is Lindelöf. Let $x \in S(\pi)$ then, there exists a $\nu = \mu_x \in I_\sigma(\mathcal{L})$ and $\pi \leq \mu_x(\mathcal{L})$ and so (**) is satisfied.

Consider the space $I_\sigma(\mathcal{L})$ and the lattice $V_\sigma(\mathcal{L})$ where $V_\sigma(\mathcal{L}) = \{V_\sigma(L); L \in \mathcal{L}\}$ and $V_\sigma(L) = \{\mu \in I_\sigma(\mathcal{L}); \mu(L) = 1\}$. It is well-known that $V_\sigma(\mathcal{L})$ is prime complete (See [1] and $V_\sigma(\mathcal{L})$ is a base for the closed sets $\tau V_\sigma(\mathcal{L})$ of $I_\sigma(\mathcal{L})$.

THEOREM 4.9. $\mathcal{L}$ satisfies (**) if and only if the topological space $I_\sigma(\mathcal{L})$, $\tau V_\sigma(\mathcal{L})$ is Lindelöf.

PROOF. i) If $I_\sigma(\mathcal{L})$, $\tau V_\sigma(\mathcal{L})$ is Lindelöf then, $V_\sigma(\mathcal{L})$ is Lindelöf and by Theorem 4.8, $V_\sigma(\mathcal{L})$ satisfies (**) and so does $\mathcal{L}$; ii) Assume that $\mathcal{L}$ satisfies (**) then, $V_\sigma(\mathcal{L})$ satisfies (**). Moreover,

$V_\sigma(\mathcal{L})$ is prime complete. Thus, by Theorem 4.6, $I_\sigma(\mathcal{L})$, $V_\sigma(\mathcal{L})$ is Lindelöf which implies that $I_\sigma(\mathcal{L})$, $\tau V_\sigma(\mathcal{L})$ is Lindelöf.

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