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ON GALOIS PROJECTIVE GROUP RINGS

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ABSTRACT. Let A be a ring with 1, C the center of A and G' an inner automorphism group of A induced by $\{U_\alpha$ in A / α in a finite group G whose order is invertible$\}$. Let $A^{G'}$ be the fixed subring of A under the action of G'. If A is a Galois extension of $A^{G'}$ with Galois group G' and C is the center of the subring $\sum_\alpha A^{G'}U_\alpha$ then $A = \sum_\alpha A^{G'}U_\alpha$ and the center of $A^{G'}$ is also C. Moreover, if $\sum_\alpha A^{G'}U_\alpha$ is Azumaya over C, then A is a projective group ring.

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1. INTRODUCTION.

Galois extensions for rings and Azumaya algebras have been intensively investigated (see References). F.R. DeMeyer ([1]) characterizes a central Galois algebra with an inner Galois group in terms of an Azumaya projective group algebra. That is, if A is a central Galois algebra over C with an inner Galois group G' induced by the units $\{U_\alpha$ / α in a finite group $G\}$, then A is a projective group algebra CG_f where f is a factor set $f(\alpha,\beta) = U_\alpha U_\beta U_{\alpha\beta}^{-1}$ for α, β in G. Conversely, if CG_f is Azumaya over C, then it is Galois over C with an inner Galois group G' induced by $\{U_\alpha\}$ ([1], Theorems 2 and 3). In the present paper, we shall study a Galois extension A over a ring $A^{G'}$ (not necessarily its center C). It will be shown that if A is Galois with Galois group G' and if the subring $\sum_\alpha A^{G'}U_\alpha$ has center C, then $A = \sum_\alpha A^{G'}U_\alpha$ such that the center of $A^{G'}$ is also C. In addition, if $A^{G'}$ is separable over C, then A becomes a projective group ring over $A^{G'}$, $A = A^{G'}G_f$ (that is, the coefficient ring C in a projective group C-algebra is replaced by a ring $A^{G'}$ as defined in [1]). In this case, A is an Azumaya C-algebra. Conversely, if $\sum_\alpha A^{G'}U_\alpha$ is an C-Azumaya C-algebra such that $\{U_\alpha\}$ are free over C, then $A = A^{G'}G_f$ and is Galois over $A^{G'}$ such that $A^{G'}$ has center C. Our results generalize the characterization for a central Galois algebra as given by F.R. DeMeyer ([1], Theorems

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2. BASIC DEFINITIONS.

Throughout, we assume that A is a ring with 1, C the center of A, G' the inner automorphism group of A induced by the set of units $\{U_\alpha$ / α in a finite group G whose order is invertible$\}$, and $A^{G'} = \{a$ in A / $\alpha'(a) = a = U_\alpha a U_\alpha^{-1}$ for each α' in $G'\}$. The projective group ring RG_f over a ring R is a ring with an R-basis $\{U_\alpha$ / α in $G\}$ such that $U_\alpha U_\beta = U_\delta f(\alpha,\beta)$ where $\alpha\beta = \delta$ in G and $rU_\alpha = U_\alpha r$ for each r in R, α, β in G and $f(\alpha,\beta) = U_\alpha U_\beta U_{\alpha\beta}^{-1}$. A separable extension T over S is a ring extension T over its subring S such that there exist elements $\{a_i, b_i$ in T, $i = 1,\dots,m$ for some integer m / $\sum aa_i\otimes b_i = \sum a_i\otimes b_i a$ for each a in T and $\sum a_i b_i = 1\}$ where $\otimes$ is over S. Such a set $\{a_i, b_i\}$ is called a separable set for T. The separable S-algebra T is a separable extension T over S which is contained in the center of T, and T is an Azumaya S-algebra if T is a separable algebra over its center S. T is a Galois extension over T^G with Galois group G if G is a finite automorphism group of T and there exist elements $\{x_i, y_i$ in T, $i = 1,\dots,k$ for some integer k / $\sum x_i y_i = 1$ and $\sum x_i\alpha(y_i) = 0$ for each $\alpha \neq 1$ in $G\}$. Such a set $\{x_i, y_i\}$ is called a Galois set for T. A Galois extension T over T^G with Galois group G is called a centralized Galois extension if T^G has the same center as T.

3. GALOIS PROJECTIVE GROUP RINGS.

Throughout, we assume that A is a ring with 1, C the center of A, G' an inner automorphism group of A induced by $\{U_\alpha$ in A / α in a finite group G whose order is invertible$\}$. We denote the set $\{a$ in A / $\alpha'(a) = a$ for each α' in $G'\}$ by $A^{G'}$ and the projective group ring over $A^{G'}$ by $A^{G'}G_f$ where $f(\alpha,\beta) = U_\alpha U_\beta U_{\alpha\beta}^{-1}$ for α, β in G. We shall generalize the characterization of a central Galois algebra as given by F.R. DeMeyer to a non-commutative case. We begin with some properties of G'.

LEMMA 3.1. (1) Let $\sum_\alpha CU_\alpha$ (or $\sum_\alpha A^{G'}U_\alpha$) be the subring generated by C (or $A^{G'}$) and $\{U_\alpha\}$ respectively. Then G' restricted to $\sum_\alpha CU_\alpha$ is isomorphic with G' restricted to $\sum_\alpha A^{G'}U_\alpha$. (2) If $\sum_\alpha A^{G'}U_\alpha$ has center C, then G' restricted to $\sum_\alpha A^{G'}U_\alpha$ is isomorphic with G'.

PROOF. Since $\alpha' = \beta'$ on $\sum_\alpha A^{G'}U_\alpha$ if and only if $U_\beta^{-1}U_\alpha$ is in the center of $\sum_\alpha A^{G'}U_\alpha$, and since $\alpha' = \beta'$ on A if and only if $U_\beta^{-1}U_\alpha$ is in C similarly, part (2) holds. Part (1) is clear.

LEMMA 3.2. If A is Galois over $A^{G'}$ with Galois group G', then $G' \cong G$.

PROOF. We first show that $\{U_\alpha$ / α in $G\}$ are free over C. In fact, let $\sum_\alpha a_\alpha U_\alpha = 0$ for a_α in C and let $\{x_i, y_i$ in A, $i = 1,\dots,m$ for some integer $m\}$ be a Galois set for A. Then $\sum x_i(\sum_\alpha a_\alpha U_\alpha)\beta^{-1}(y_i)$

$= 0 = \sum_\alpha a_\alpha(x_i\alpha\beta^{-1}(y_i))U_\alpha = a_\beta U_\beta$ for any β in G. Next, let α', β' be in G' such that $\alpha'(a) = \beta'(a)$ for all a in A. Then $U_\beta^{-1}U_\alpha$ is in C. Thus there is an c in C such that $U_\beta^{-1}U_\alpha = c$; and so $U_\alpha = cU_\beta$. But $\{U_\alpha\}$ are free over C, so $U_\alpha = U_\beta$. Thus $\alpha = \beta$. Moreover, the map $\alpha \to \alpha'$ from G' to G is clearly a group homomorphism, so $G' \cong G$.

THEOREM 3.3. Let A be a Galois extension over $A^{G'}$ with Galois group G'. If $\sum_\alpha A^{G'}U_\alpha$ has center C, then $A = \sum_\alpha A^{G'}U_\alpha$ which is generated by $A^{G'}$ and $\{U_\alpha\}$.

PROOF. As given in the proof of Lemma 3.2, $\{U_\alpha\}$ are free over C, so there exists a projective group subalgebra CG_f of A where $f(\alpha,\beta) = U_\alpha U_\beta U_{\alpha\beta}^{-1}$ for α, β in G. Since the order of G is invertible, $(1/n)\sum_\alpha U_\alpha \otimes U_\alpha^{-1}$ is a separable element for A where n is the order of G. Hence CG_f is a separable projective group algebra over C. Now, let r be an element in the center of CG_f. Then $rU_\alpha = U_\alpha r$ for each α in G. On the other hand, for any t in $A^{G'}$, $U_\alpha tU_\alpha^{-1} = t$ for each α in G, so $U_\alpha t = tU_\alpha$. Hence $rt = tr$ (for r is in CG_f). Since $CG_f \subset \sum_\alpha A^{G'}U_\alpha$, r is in the center of $\sum_\alpha A^{G'}U_\alpha$. By hypothesis, $\sum_\alpha A^{G'}U_\alpha$ has center C, so r is in C. Thus the center of CG_f is contained in C. Clearly, C is contained in the center of CG_f, so CG_f has center C. Therefore, CG_f is an Azumaya algebra over C. But then CG_f is a central Galois algebra over C with Galois group G' restricted to CG_f by Theorem 3 in [1]. Since $CG_f \subset \sum_\alpha A^{G'}U_\alpha$ such that G' restricted to $\sum_\alpha A^{G'}U_\alpha$ is isomorphic with G' restricted to CG_f by Lemma 3.1, $\sum_\alpha A^{G'}U_\alpha$ is Galois over $A^{G'}$ with Galois group G' restricted to $\sum_\alpha A^{G'}U_\alpha$ (for the Galois set for CG_f is also a Galois set for $\sum_\alpha A^{G'}U_\alpha$). Moreover, let $\{x_i, y_i \text{ in } \sum_\alpha A^{G'}U_\alpha \;/\; i = 1,\ldots,k\}$ be a Galois set for $\sum_\alpha A^{G'}U_\alpha$ over $A^{G'}$. By hypothesis, A is Galois over $A^{G'}$ with Galois group G', so $G' \cong G'$ restricted to $\sum_\alpha A^{G'}U_\alpha$ by Lemma 3.2. Thus $\{x_i, y_i\}$ is also a Galois set for A over $A^{G'}$. Therefore, A is finitely generated and projective over $A^{G'}$ with a dual basis $\{x_i, (tr)y_i\}$ where $tr = \sum_\alpha \alpha$ (the trace of G) ([2], see the proof of Theorem 1, P. 119) Hence, for any a in A, $a = \sum x_i tr(y_i a)$ which is in $\sum_\alpha A^{G'}U_\alpha$. Thus $A = \sum_\alpha A^{G'}U_\alpha$.

We recall that a Galois extension T over T^G with Galois group G is called a centralized Galois extension if A^G has the same center as A.

COROLLARY 3.4. By keeping the hypotheses of Theorem 3.3, A is a centralized Galois extension.

PROOF. Clearly, C is contained in the center of $A^{G'}$. Conversely, for any r in the center of $A^{G'}$, r is in the center of $\sum_\alpha A^{G'}U_\alpha$. By Theorem 3.3, $A = \sum_\alpha A^{G'}U_\alpha$, so r is in C. Thus the center of $A^{G'}$ is also C.

Next is a condition under which A is a projective group ring.

THEOREM 3.5. By keeping the hypotheses and notations of Theorem 3.3, if $\sum_\alpha A^{G'}U_\alpha$ is separable over C, then $A = A^{G'}G_f$ such that $A^{G'}$ is an Azumaya C-algebra.

PROOF. By Theorem 3.3, $A = \sum_\alpha A^{G'}U_\alpha$ containing an Azumaya algebra CG_f over C. By hypothesis, A is an Azumaya C-algebra containing an Azumaya subalgebra CG_f, so $A \cong Z_A(CG_f)\otimes_C CG_f$ where $Z_A(CG_f)$ is the commutant of CG_f in A such that $Z_A(CG_f)$ is an Azumaya C-subalgebra by the commutant theorem for Azumaya algebras ([3], Theorem 4.3, P. 57). Noting that $Z_A(CG_f) = A^{G'}$, we conclude that $A \cong A^{G'}\otimes_C CG_f \cong A^{G'}G_f$ such that $A^{G'}$ is an Azumaya C-algebra.

The following is a property of a Galois projective group ring.

THEOREM 3.6. Let R be a ring with 1 (not necessarily commutative). If RG_f is Galois over $(RG_f)^{G'}$ with Galois group G', then $(RG_f)^{G'} = R$.

PROOF. Let C be the center of RG_f. Then $C \subset (RG_f)^{G'}$. Since RG_f is Galois, $\{U_\alpha\}$ are free over C by the proof of Lemma 3.2. Hence there exists a projective group subalgebra CG_f of RG_f. Next, let D be the center of R. We claim that $C = D$. In fact, for any $x = \sum_\alpha r_\alpha U_\alpha$ in C and t in R, $xt = tx$, so r_α are in the center of R. But then $C \subset DG_f$. Hence $CG_f \subset DG_f$. Since $\{U_\alpha\}$ are free over C and D respectively, $C \subset D$. Clearly, $D \subset C$, so $C = D$. Thus the center of CG_f is C. Noting that the order of G is invertible we conclude that CG_f is an Azumaya C-algebra; and so CG_f is a central Galois C-algebra with Galois group G' ([1], Theorem 3). Moreover, since $RG_f \cong R\otimes_C CG_f$ such that C is a C-direct summand of CG_f, there exists an element d in CG_f such that $tr(d) = 1$ by using the fact that $\mathrm{Hom}_{A^{G'}}(A, A^{G'}) \cong (tr)A$ (see [2], P. 119 and the introduction to Section 2 in [5]). Then, for any $\sum_\alpha r_\alpha \otimes U_\alpha$ in $(RG_f)^{G'}$ and β in G, $\beta(\sum_\alpha r_\alpha \otimes U_\alpha) = \sum_\alpha r_\alpha \otimes \beta(U_\alpha) = \sum_\alpha r_\alpha \otimes \beta(U_\alpha)tr(d) = \sum_\alpha r_\alpha \otimes tr(U_\alpha d)$ which is in R (for $tr(U_\alpha d)$ is in C). Thus $(RG_f)^{G'} = R$.

We now generalize the theorem of F.R. DeMeyer that if KG_f is Azumaya over a commutative ring K, then KG_f is Galois over K with Galois group G' ([1], Theorem 3).

THEOREM 3.7. If $\sum_\alpha A^{G'}U_\alpha$ is an Azumaya C-algebra with $\{U_\alpha\}$ a free set over C, then $\sum_\alpha A^{G'}U_\alpha = A^{G'}G_f$ and is a centralized Galois extension over $A^{G'}$ with Galois group G'.

PROOF. Since $\{U_\alpha\}$ are free over C, $\sum_\alpha A^{G'}U_\alpha$ contains a projective group algebra CG_f. Since the order of G is invertible, CG_f is separable over C. Since the center of CG_f is in the center of $\sum_\alpha A^{G'}U_\alpha$, it is equal to C. Hence CG_f is an Azumaya C-algebra. Thus CG_f is a central Galois C-algebra with Galois group G' ([1], Theorem 3, and Lemma 3.1). Noting that $CG_f \subset \sum_\alpha A^{G'}U_\alpha$, we conclude that $\sum_\alpha A^{G'}U_\alpha$ is Galois over $A^{G'}$ with Galois group G'. Moreover, since $\sum_\alpha A^{G'}U_\alpha$ is Azumaya over C by hypothesis, $\sum_\alpha A^{G'}U_\alpha = A^{G'}G_f$ by the commutant theorem for Azumaya algebras as given in the proof of Theorem 3.5. Thus the proof is complete.

COROLLARY 3.8. Let R be a ring with 1 and C the center of the projective group ring RG_f over R. If RG_f is an Azumaya algebra such that $\{U_\alpha\}$ are free over C, then RG_f is a centralized Galois extension over R with Galois group G'.

PROOF. Let the center of R be D. Then there exists a projective group algebra DG_f in RG_f. Clearly, $RG_f \cong R \otimes_D DG_f$. Since $\{U_\alpha\}$ are free over C by hypothesis, we can show that C = D by a similar proof of Theorem 3.6. Moreover, since $aU_\alpha = U_\alpha a$ for each a in R and α in G, the center of DG_f is contained in C. Clearly, C is contained in the center of DG_f, so C = the center of DG_f. Hence D = the center of DG_f. Thus DG_f is a central Galois D-algebra. Therefore RG_f is Galois over R by Theorem 3.6 such that the center of R is C.

We conclude the paper with two more properties of a Galois projective group ring RG_f.

THEOREM 3.9. Let RG_f be a Galois projective group ring with Galois group G' over a ring R and with center C. Then (1) the centralizer of the projective group algebra CG_f in RG_f is R, and (2) the center of R is equal to C (and hence RG_f is a centralized Galois extension over R).

PROOF. (1) By Theorem 3.6, $R = (RG_f)^{G'}$. Since G' is an inner automorphism group of RG_f induced by $\{U_\alpha\}$, part (1) is immediate. (2) Let D be the center of R. Then it is easy to verify that DG_f = the centralizer of R in RG_f. By part (1), $DG_f = CG_f$. Since RG_f is Galois, $\{U_\alpha\}$ are free over C by the proof of Lemma 3.2. But then D = C.

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