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SOME BAZILEVIČ FUNCTIONS OF ORDER BETA

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ABSTRACT. Distortion theorems and coefficient estimates are obtained for a new class of Bazilevič functions.

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1. INTRODUCTION.

Let S be the class of normalized functions regular and univalent in the unit disc $D = z : \{|z| < 1\}$ and S^* the subclass of starlike functions. Denote by $P(\beta)$, the class of functions which are regular in D and such that for $h \in P(\beta)$, $h(0) = 1$ and $\mathrm{Re}\, h(z) > \beta$ for $z \in D$. We write $P = P(0)$.

Bazilevič [1] showed that the class of normalized regular functions f with representation

$$f(z) = \left(\alpha\int_0^z p(t)\, g(t)^{\alpha}\, t^{-1}\, dt\right)^{\frac{1}{\alpha}} \tag{1.1}$$

when $\alpha > 0$, $g \in S^*$ and $p \in P$ for $z \in D$ forms a subclass of S. We denote this class of functions by $B(\alpha)$. See also [2].

Let $\alpha > 0$. Then it follows easily from (1.1) that $f \in B(\alpha)$ if, and only if, there exists $g \in S^*$ such that for $z \in D$

$$\mathrm{Re}\,\frac{z\, f'(z)}{f(z)^{1-\alpha} g(z)^{\alpha}} > 0. \tag{1.2}$$

In [3], Singh considered the subclass $B_1(\alpha)$ of $B(\alpha)$ obtained by taking $g(z) \equiv z$ in (1.2). Thus $f \in B_1(\alpha)$ if , and only if, for $\alpha > 0$ and $z \in D$

$$\operatorname{Re} \frac{z^{1-\alpha} f'(z)}{f(z)^{1-\alpha}} > 0.$$

We extend this class of functions as follows:

DEFINITION. Let f be regular in D with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \, . \tag{1.3}$$

Then if $\alpha > 0$ and $0 \leq \beta < 1$, $f \in B_1(\alpha,\beta)$ if, and only if, for $z \in D$

$$\operatorname{Re} \frac{z^{1-\alpha} f'(z)}{f(z)^{1-\alpha}} > \beta. \tag{1.4}$$

We note that $B_1(1,0) = R$, the class of functions whose derivative has real part [4]. $B_1(1,\beta)$ was considered in [5]. Zamorski [6] and Thomas [7] solved the coefficient problem for $f \in B(\frac{1}{N})$, in the case when N is a positive integer. In [7], sharp distortion theorems were obtained for $f \in B_1(\alpha)$ for $\alpha > 0$. The object of this paper is to extend these results to the class $B_1(\alpha,\beta)$. The class $B_1(\alpha,\beta)$ has also recently been considered in [8].

2. RESULTS.

Distortion Theorems

THEOREM 1. Let $f \in B_1(\alpha,\beta)$. Then for $z = re^{i\theta} \in D$, $0 \leq r < 1$,

(i) $Q_2(r)^{\frac{1}{\alpha}} \leq |f(z)| \leq Q_1(r)^{\frac{1}{\alpha}}$,

(ii) if $0 < \alpha \leq 1$,

$$r^{\alpha-1} Q_2(r)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-r)(1-\beta)}{(1+r)} + \beta\right) \leq |f'(z)| \leq r^{\alpha-1} Q_1(r)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1+r)(1-\beta)}{(1-r)} + \beta\right)$$

and if $\alpha \geq 1$

$$r^{\alpha-1} Q_1(r)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-r)(1-\beta)}{(1+r)} + \beta\right) \leq |f'(z)| \leq r^{\alpha-1} Q_2(r)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1+r)(1-\beta)}{(1-r)} + \beta\right)$$

where

$$Q_1(r) = \alpha\int_0^r \rho^{\alpha-1} \left(\frac{(1+\rho)(1-\beta)}{(1-\rho)} + \beta\right) d\rho \, ,$$

and

$$Q_2\ (r) = \alpha\int_0^r \rho^{\alpha-1}\ \left(\frac{(1-\rho)(1-\beta)}{(1+\rho)} + \beta\right)d\rho\ .$$

Equality holds in all cases for the function f_ϕ , defined by

$$f_\phi\ (z) = \left(\alpha\int_0^z t^{\alpha-1}\ \left(\frac{(1+te^{i\phi})(1-\beta)}{(1-te^{i\phi})} + \beta\right)\ dt\right)^{\frac{1}{\alpha}} \tag{2.1}$$

where $\phi = 0$ or π .

PROOF.

(i) Since $f \in B_1(\alpha,\beta)$, and it follows from (1.4) that

$$(1-\beta)p(z) = \frac{z^{1-\alpha}f'(z)}{f(z)^{1-\alpha}} - \beta$$

for $z \in D$ and $p \in P$.
Thus

$$f(z)^\alpha = \alpha\int_0^z t^{\alpha-1}\ (p(t)(1-\beta) + \beta)dt \tag{2.2}$$

and since $|p(z)| < \frac{1+r}{1-r}$ for $z \in D$, (see eg. [9]),

$$|f(z)|^\alpha < \alpha\int_0^r \rho^{\alpha-1}\ \left(\frac{(1+\rho)(1-\beta)}{1-\rho} + \beta\right)d\rho$$

$$= Q_1(r).$$

To obtain the left-hand inequality in (i), write

$$h(z) = \frac{z^{1-\alpha}f'(z)}{f(z)^{1-\alpha}}\ . \tag{2.3}$$

Then (1.4) shows that $h \in p(\beta)$. Thus from, [5] (Theorem 1 with $c=1-2\beta$ and $n=1$), we obtain

$$\frac{(1-r)(1-\beta)}{(1+r)} + \beta < |h(z)| < \frac{(1+r)(1-\beta)}{(1-r)} + \beta\ . \tag{2.4}$$

Hence from (2.3) and (2.4) we have

$$\left|\frac{d}{dz}\ [f(z)]^\alpha\right| > \alpha r^{\alpha-1}\ \left(\frac{(1-r)(1-\beta)}{(1+r)} + \beta\right). \tag{2.5}$$

Now let z_1 , $|z_1| = r$ be chosen so that $|f(z_1)^\alpha| < |f(z)^\alpha|$ for all z with $|z| = r$. Writing $\omega = f(z_1)^\alpha$, it follows that since f is univalent, the line segment λ from 0 to ω lies entirely in the image of D. Let ℓ be the pre-image of λ , then by (2.5)

$$|f(z)|^{\alpha} \geq |f(z_1)|^{\alpha} = \int_{\lambda} |d\omega| = \int_L \left|\frac{d\omega}{dz_1}\right| |dz_1|$$

$$\geq \int_0^r \alpha\rho^{\alpha-1} \left(\frac{(1-\rho)(1-\beta)}{(1+\rho)} + \beta\right) d\rho$$

which is the left-hand inequality in (i).

(ii) From (2.3) we have for $z = re^{i\theta}$

$$|f'(z)| = r^{\alpha-1} |f(z)|^{1-\alpha} |h(z)| \tag{2.6}$$

if $0 < \alpha \leq 1$, the inequalities follow at once from (2.6), (2.4) and (i). If $\alpha \geq 1$, (i) gives

$$Q_1(r)^{\frac{1-\alpha}{\alpha}} \leq |f(z)|^{1-\alpha} \leq Q_2(r)^{\frac{1-\alpha}{\alpha}} . \tag{2.7}$$

Applying (2.4) and (2.7) to (2.6) gives the required result. Equality is attained in and (i) for f_0 and in (ii) for f_0 when $0 < \alpha \leq 1$ and for f_π when $\alpha \geq 1$.

The following shows that as $\alpha \to 0$ the bounds in Theorem 1 are asymptotic to the distortion theorems for starlike functions of order $\beta > 0$ (see eg. [9]).

THEOREM 2. For $0 \leq r < 1$, let $Q_1(r)$ and $Q_2(r)$ be defined as in Theorem 1. Then as $\alpha \to 0$

(i) $Q_1(r)^{\frac{1}{\alpha}} \sim \dfrac{r}{(1-r)^{2(1-\beta)}}$,

(ii) $Q_2(r)^{\frac{1}{\alpha}} \sim \dfrac{r}{(1+r)^{2(1-\beta)}}$,

(iii) $Q_1(r) \sim Q_2(r) \sim 1$.

PROOF.

We prove (i), since (ii) and (iii) are similar.

As $\alpha \to 0$,

$$Q_1(r)^{\frac{1}{\alpha}} = \alpha\int_0^r \rho^{\alpha-1} \left(\frac{(1+\rho)(1-\beta)}{1-\rho} + \beta\right)d\rho$$

$$= r\left(1+ 2\alpha(1-\beta)r^{-\alpha} \int_0^r \frac{\rho^{\alpha}}{1-\rho}\, d\rho\right)^{\frac{1}{\alpha}}$$

$$\sim r(1-2\alpha(1-\beta)r^{-\alpha} \log(1-r))^{\frac{1}{\alpha}}$$

$$\sim re^{-2(1-\beta)\log(1-r)} = \frac{r}{(1-r)^{2(1-\beta)}} .$$

COROLLARY.

Suppose that $f(z) \neq \omega$ for $z \in D$, then

$$|\omega| > Q_2(1)^{\frac{1}{\alpha}} \sim 4^{\beta-1} \text{ as } \alpha \to 0.$$

PROOF.

Let $\alpha > 0$, and ω be a point on the boundary of $f(D)$ closest to the origin. Let L_1 denote the straight line from 0 to ω and L its pre-image in D. Then $|\omega| > |F(z)|$ for $z \in L \cap D$. Since the circle $|z| = r$ intersects L, at least once, Theorem 1 (i) gives $|\omega| > Q_2(r)^{\frac{1}{\alpha}}$.

Thus Theorem 2 (ii) gives

$$|\omega| > Q_2(1)^{\frac{1}{\alpha}} \sim 4^{\beta-1} \text{ as } \alpha \to 0.$$

3. A COEFFICIENT THEOREM

Notation: $\sum_{n=0}^{\infty} \alpha_n z^n \ll \sum_{n=0}^{\infty} \beta_n z^n$ means $|\alpha_n| < |\beta_n|$ for $n > 0$.

THEOREM 3. Let $f \in B_1(\frac{1}{N},\beta)$ and be given by (1.3) where N is a positive integer. Suppose also that for $z \in D$,

$$f_0(z) = z + \sum_{n=0}^{\infty} \gamma_n z^n \text{ where } f_0(z) \text{ is given by (2.1).}$$

Then (i) $f(z) \ll f_0(z)$,

and (ii) $\gamma_n \sim \left(\frac{2(1-\beta)}{N}\right)^N \left(\frac{N}{n}\right) (\log n)^{N-1}$ as $n \to \infty$.

PROOF.

(i) Thomas [7], proved that if $|\alpha_n| < |\beta_n|$, then for m = 1, 2, 3, ...,

$$\left(\sum_{n=1}^{\infty} \alpha_n z^n\right)^m \ll \left(\sum_{n=1}^{\infty} \beta_n z^n\right)^m.$$

Write $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$. Then (2.2) gives

$$f(z)^{\frac{1}{N}} = \frac{1}{N} \int_0^z t^{\alpha-1}\left[\left(1 + \sum_{k=1}^{\infty} p_k t^k\right)(1+\beta) + \beta\right]dt$$

$$= \frac{1}{N}\left[N(1-\beta)z^{\frac{1}{N}} + (1-\beta)\sum_{k=1}^{\infty}\left(\frac{p_k z^{k+\frac{1}{N}}}{k+\frac{1}{N}}\right) + \beta N z^{\frac{1}{N}}\right]$$

$$= z^{\frac{1}{N}}\left(1 + \frac{(1-\beta)}{N}\sum_{k=1}^{\infty}\frac{p_k z^k}{(k+\frac{1}{N})}\right).$$

Thus

$$f(z) = z\left(1 + \frac{(1-\beta)}{N}\sum_{k=1}^{\infty}\frac{p_k z^k}{(k+\frac{1}{N})}\right)^N$$

and since $p \in P$, we have $|P_k| \le 2$ [6]. Hence

$$f(z) = z\left(1+\frac{(1-\beta)}{N}\sum_{k=1}^{\infty}\frac{P_k z^k}{(k+\frac{1}{N})}\right)^N \ll z\left(1+\frac{(1-\beta)}{N}\sum_{k=1}^{\infty}\frac{2z^k}{(k+\frac{1}{N})}\right)^N = f_0(z).$$

(ii) Putting $\alpha = \frac{1}{N}$ in (2.1), we have

$$f_0(z) = z + \sum_{n=2}^{\infty}\gamma_n z^n = z\left(1 + \frac{2(1-\beta)}{N}\sum_{n=1}^{\infty}\frac{z^n}{(n+\frac{1}{N})}\right)^N$$

$$= z\sum_{\nu=0}^{\infty}\binom{N}{\nu}\left(\frac{2(1-\beta)}{N}\right)^{\nu}\left(\sum_{n=1}^{\infty}\frac{z^n}{(n+\frac{1}{N})}\right)^{\nu}.$$

Let

$$\left(\sum_{n=1}^{\infty}\frac{z^n}{(n+\frac{1}{N})}\right)^{\nu} = \sum_{n=\nu}^{\infty} D_n(\nu) z^n \qquad (\nu = 0, 1, 2, 3, \ldots).$$

Thomas [7] proved that $D_n^{(\nu)} \sim \frac{\nu}{N}(\log n)^{\nu-1}$ as $n \to \infty$ and so this gives

$$\gamma_n = \sum_{\nu=0}^{\infty}\binom{N}{\nu}\left(\frac{2(1-\beta)}{N}\right)^{\nu} D_n^{(\nu)}$$

$$\sim \left(\frac{2(1-\beta)}{N}\right)^N\binom{N}{n}(\log n)^{N-1} \text{ as } n \to \infty.$$

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