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SOME SUFFICIENT CONDITIONS FOR UNIVALENCE

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ABSTRACT. A new subclass $R(\alpha)$, $0 \leq \alpha \leq 1$, of the class $S_t(1/2)$ - the class of starlike functions of order 1/2 - is introduced and it is shown that $R(\alpha)$ is closed with respect to the Hadamard product of analytic functions. Some sufficient conditions for the normalized regular functions to be univalent in the unit disk E are given.

KEY WORDS AND PHRASES. Convex function, close-to-convex function, starlike function of order 1/2, univalent function, Hadamard product.
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1. INTRODUCTION.

Let A denote the class of functions $f(z) = z + a_2 z^2 + \ldots$ which are regular in the unit disk $E = \{z/|z| < 1\}$. We denote by S the subclass of A consisting of functions f which are univalent in E, K will stand for the usual subclass of S whose members are convex in E. A function $f \in A$ is said to be close-to-convex in E if and only if $\mathrm{Re}(f'(z)/g'(z)) > 0$, $z \in E$, for some $g \in K$. Since $g(z) \equiv z$ is convex in E, the functions $f \in A$ which satisfy $\mathrm{Re}\, f'(z) > 0$, $z \in E$ are close-to-convex in E. It is well known that every close-to-convex function in E is univalent in E. For a given α, $0 \leq \alpha < 1$, denote by $S_t(\alpha)$ the subclass of S consisting of functions f which satisfy the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in E.$$

$S_t(\alpha)$ is called the class of starlike functions of order α. It is also well known that for $0 \leq \alpha \leq \beta < 1$, $S_t(\beta) \subseteq S_t(\alpha)$.

In the present paper we introduce a new subclass $R(\alpha)$ of the class $S_t(1/2)$ and prove that $R(\alpha)$ is closed with respect to convolution/Hadamard product of analytic functions. Some sufficient conditions are given for a function $f \in A$ to be in the class S.

2. PRELIMINARIES.

We shall need the following definitions and results. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are regular in E, then their convolution/Hadamard product is the function denoted by $f * g$ and defined by the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \tag{2.1}$$

Let a,b and c be any complex numbers with c neither zero nor a negative integer. Then the hypergeometric function $F(a,b;c;z)$ is defined in Rainville [1, p. 45] by

$$F(a,b;c;z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} z^n, \tag{2.2}$$

where $(\mu)_n$ is the Pochhammer symbol defined by

$$(\mu)_n = \begin{cases} 1, \text{ if } n = 0 \\ \mu(\mu+1)\dots(\mu+n-1), \text{ if } n \in N = \{1,2,3,\dots\}. \end{cases} \tag{2.3}$$

It is known that the series on the right in (2.2) is convergent for $z \in E$.

Now we define the function $\varphi(a,c)$ by

$$(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \quad (c \neq 0,-1,-2,\dots;\ z \in E). \tag{2.4}$$

From (2.2) and (2.4) we immediately get

$$(a,c;z) = zF(1,a;c;z) \tag{2.5}$$

LEMMA 2.1. [1, p. 47]. If $|z| < 1$ and if $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$,

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt \tag{2.6}$$

LEMMA 2.2. For a given real number α, let

$f_\alpha(z) = \sum_{n=1}^{\infty} n^{-\alpha} z^n$, $z \in E$. Then f_α is convex whenever $\alpha > 0$.

LEMMA 2.3. Let $f \in S_t(1/2)$ and $g \in S_t(\beta)$, where $1/2 < \beta < 1$. Then $f * g$ is a member of $S_t(\beta)$.

LEMMA 2.2 is due to Lewis [2] and Lemma 2.3 follows the Corollary 1 in Lewis [3] by taking $\alpha = 1/2$.

LEMMA 2.4. If $f \in K$, then $\mathrm{Re}(f(z)/z) > 1/2$, $z \in E$.

LEMMA 2.5. If $p(z)$ is analytic in E, $p(0) = 1$ and $\text{Re}\, p(z) > 1/2$, $z \in E$, then for any function F, analytic in E, the function $P * F$ takes values in the convex hull of $F(E)$. Lemma 2.4. is due to Strohhäcker [4] and the assertion of Lemma 2.5 readily follows by using Herglotz' representation for $P(z)$.

3. THEOREMS AND THEIR PROOFS.

For $0 < \alpha < 1$, let $R(\alpha)$ denote the class of functions $f \in A$ which satisfy the condition

$$\sum_{n=1}^{\infty} n^{\alpha} z^n * f(z) \in S_t(\tfrac{1+\alpha}{2}), \quad z \in E. \tag{3.1}$$

Clearly $R(0) = S_t(1/2)$ and $f \in R(1)$ if and only if $f(z) \equiv z$.

THEOREM 3.1. (i) If $0 < \alpha < \beta < 1$, then $R(\beta) \subseteq R(\alpha)$. (ii) $R(\gamma)$ is a subclass of $S_t(1/2)$ for every $\gamma > 0$.

PROOF. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R(\beta)$ so that

$$g(z) = \sum_{n=1}^{\infty} n^{\beta} z^n * f(z) \in S_t((1+\beta)/2). \tag{3.2}$$

Now

$$\sum_{n=1}^{\infty} n^{\alpha} z^n * f(z) = \Big(\sum_{n=1}^{\infty} n^{\beta} z^n * f(z)\Big) * \sum_{n=1}^{\infty} n^{\alpha-\beta} z^n$$

$$= g(z) * k(z), \tag{3.3}$$

where $k(z) = \sum_{n=1}^{\infty} n^{-(\beta-\alpha)} z^n$.

Since $\beta-\alpha > 0$, therefore by Lemma 2.2, $k(z) \in K \subseteq S_t(1/2)$. In view of Lemma 2.3, we now get from (3.2) and (3.3) that

$$g * k \in S_t((1+\beta)/2) \subseteq S_t((1+\alpha)/2), \qquad (\text{as } \alpha < \beta).$$

Hence from (3.3) and (3.1) we conclude that $f \in R(\alpha)$. This completes the proof of part (i). The proof of part (ii) follows immediately from part (i) and from the observation that $R(0) = S_t(1/2)$.

THEOREM 3.2. If f and g both belong to $R(\alpha)$, then $f * g$ also belongs to $R(\alpha)$.

PROOF. Since $f \in R(\alpha)$, therefore

$$h(z) = \sum_{n=1}^{\infty} n^{\alpha} z^n * f(z) \quad \in \; S_t((1+\alpha)/2). \tag{3.4}$$

Now

$$\sum_{n=1}^{\infty} n^{\alpha} z^n * (f * g)(z) = \Big(\sum_{n=1}^{\infty} n^{\alpha} z^n * f(z)\Big) * g(z)$$

$$= h(z) * g(z). \tag{3.5}$$

Since $g \varepsilon R(\alpha) \subseteq S_t(1/2)$, therefore in view of Lemma 2.3, we get, from (3.4) and (3.5) that

$$h * g \varepsilon S_t((1+\alpha)/2),$$

which in turn implies that

$$f * g \varepsilon R(\alpha).$$

This completes the proof of our theorem.

THEOREM 3.3. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \varepsilon A$ and satisfies the condition

$$\text{Re}\left[1 + \sum_{n=2}^{\infty} n^{\alpha} a_n z^{n-1}\right] > 0, \ \alpha > 1, \ z \varepsilon E, \tag{3.6}$$

then $\text{Re}\, f'(z) > 0$, $z \varepsilon E$. Hence $f(z)$ is close-to-convex in E and therefore univalent in E.

PROOF. We can write

$$f'(z) = 1 + \sum_{n=2}^{\infty} n\, a_n z^{n-1} = (1 + \sum_{n=2}^{\infty} n^{\alpha} a_n z^{n-1}) * (1 + \sum_{n=2}^{\infty} \frac{z^{n-1}}{n^{\alpha-1}}) \ldots \tag{3.7}$$

Now by Lemma 2.2, the function $k_{\alpha}(z) = z + \sum_{n=2}^{\infty} (z^n/n^{\alpha-1})$ is convex for $\alpha > 1$. Therefore, in view of Lemma 2.4,

$$\text{Re}\, \frac{k_{\alpha}(z)}{z} = \text{Re}\left[1 + \sum_{n=2}^{\infty} \frac{z^{n-1}}{n^{\alpha-1}}\right] > 1/2. \tag{3.8}$$

Thus, from (3.6), (3.7), (3.8) and Lemma 2.5, we conclude that $\text{Re}\, f'(z) > 0$.

THEOREM 3.4. Let $f \varepsilon A$ and let for $0 < \beta < \alpha$, the condition

$$\text{Re}\left[(\varphi(\alpha,\beta;z) * f(z))'\right] > 1/2, \quad z \varepsilon E, \tag{3.9}$$

be satisfied. Then $\text{Re}\, f'(z) > 0$, $z \varepsilon E$. Hence $f(z)$ is close-to-convex in E and therefore univalent in E.

PROOF. The case when $\alpha = \beta$ is obvious, therefore we let $\beta < \alpha$. We can write

$$f'(z) = \left[\frac{\varphi(\alpha,\beta;z)}{z} * f'(z)\right] * \left[\frac{\varphi(\beta,\alpha;z)}{z}\right]$$

$$=(\varphi(\alpha,\beta;z) * f(z))' * \left[\frac{\varphi(\beta,\alpha;z)}{z}\right]. \tag{3.10}$$

Now from (2.5) and Lemma 2.1, we have

$$\frac{\varphi(\beta,\alpha;z)}{z} = F(1,\beta;\alpha;z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\,\Gamma(\alpha-\beta)} \int_0^1 t^{(\beta-1)}(1-t)^{\alpha-\beta-1}(1-tz)^{-1}\, dt.$$

Since Re $\left[t^{\beta-1}(1-t)^{\alpha-\beta-1}(1-tz)^{-1}\right] > 0$ for all t, $0 < t < 1$ and for all z, $z \varepsilon E$, it follows that

$$\operatorname{Re}\left[\frac{\varphi(\beta,\alpha;z)}{z}\right] > 0,\ z \varepsilon E. \tag{3.11}$$

Form (3.9), (3.10), (3.11) and Lemma 2.5 the assertion of the theorem now follows.

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