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# SOME RESULTS ON THE SPAN OF FAMILIES OF BANACH VALUED INDEPENDENT, RANDOM VARIABLES

**ROHAN HEMASINHA**

University of West Florida
Pensacola, FL 32514

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ABSTRACT. Let E be a Banach space, and let $(\Omega, \mathcal{F}, P)$ be a probability space. If $L^1(\Omega)$ contains an isomorphic copy of $L^1[0,1]$ then in $L^p_E(\Omega)$ $(1 \leq p < \infty)$, the closed linear span of every sequence of independent, E valued mean zero random variables has infinite codimension. If E is reflexive or B-convex and $1 < p < \infty$ then the closed (in $L^p_E(\Omega)$) linear span of any family of independent, E valued, mean zero random variables is super-reflexive.

KEY WORDS AND PHRASES. Banach valued random variable, Unconditional basic sequence, Finite representability, Super-reflexive banach space, B-convex banach space, $(n,\delta)$-tree.
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## 1. INTRODUCTION.

Linear spans of sequences of independent real variables have been studied by several authors. In [1] and [2] H.P. Rosenthal utilized their properties to define a new class of Banach spaces that has had important applications in the structure theory of Banach spaces. In this paper we consider subspaces spanned by Banach valued random variables.

Our notation is the following. $(\Omega, \mathcal{F}, P)$ denotes a probability space. E is a Banach space and $\mu$ denotes expectation. A Banach valued random variable is Bochner measurable $x: \Omega \to E$ and for $1 \leq p < \infty$, $L^p_E(\Omega)$ is the space of E valued random variables x for which $\mu(||x||^p) < \infty$.

We first state and prove a set of assertions (Lemmas 1.1 and 1.2) regarding Banach valued random variables whose scalar counterparts are well known. As a consequence, we obtain that any sequence of independent, mean zero, E valued random variables, form an unconditional Basic sequence in $L^p_E(\Omega)$. This enables us to show that in $L^p_E(\Omega)$ the closed linear span of any sequence of mean zero random variables has infinite codimension (Theorem 1.5). Furthermore, using a characterization (due to R.C. James) of super-reflexivity by infinite trees we show that when E is reflexive or B-convex, then in $L^p_E(\Omega)$ $(1 < p < \infty)$ the closed subspace spanned by any family of mean zero random variable is super-reflexive.

LEMMA 1.1. Let $x,y: \Omega \to E$ be two independent, mean zero random variables. Then
$$\mu||y||^r,\ \mu||x||^r \leq \mu||x+y||^r \qquad 1 \leq r < \infty$$

PROOF. Let $P_x$, $P_y$ denote the distribution functions of $x,y$ respectively. Then

$$\begin{aligned}\mu||x+y||^r &= \iint ||x+y||^r \, dP_x(x)dP_y(y) \quad \text{(By independence)}\\ &= \int dP_y(y) \int ||x+y||^r dP_x(x)\\ &\geq \int dP_y(y) \left(\int ||x+y|| dP_x(x)\right)^r\\ &\geq \int dP_y(y) \left\|\int (x+y)dP_x(x)\right\|^r\\ &= \int dP_y(y) \left\|\int x dP_x + y\right\|^r\\ &= \int ||y||^r dP_y(y) \qquad \left(\int x dP_x(x) = 0 \text{ since } \mu(x) = 0\right)\\ &= \mu(||y||^r)\end{aligned}$$

Similarly $\mu(||x||^r) \leq \mu(||x+y||^r)$.

LEMMA 1.2. Assume $X_1, \ldots X_n$ are independent, E valued, mean zero random variables. Let $\{\varepsilon_i\}_{1 \leq i \leq n}$ be any choice of signs.

Then for $1 \leq p < \infty$ $\int ||\varepsilon_1 X_1 + \ldots + \varepsilon_n X_n||^p dP \leq 2^p \int ||X_1 + \ldots + X_n||^p dP$

PROOF. For any choice of signs $\{\varepsilon_i\}_{1 \leq i \leq n}$ let $S_1 = \{i | \varepsilon_i = +1\}$, $S_2 = \{i | \varepsilon_i = -1\}$.

Then $\sum_{i \in S_1} \varepsilon_i X_i - \sum_{i \in S_2} \varepsilon_i X_i = \sum_{i=1}^n X_i$ and $\sum_{i \in S_1} \varepsilon_i X_i + \sum_{i \in S_2} \varepsilon_i X_i = \sum_{i=1}^n \varepsilon_i X_i$.

Set $X = \sum_{i \in S_1} \varepsilon_i X_i$, $Y = \sum_{i \in S_2} \varepsilon_i X_i$.

Then X,Y are independent. Therefore by Lemma 1.1, $(\mu||X||^p)^{1/p} \leq (\mu||X-Y||^p)^{1/p}$

$(\mu||Y||^p)^{1/p} \leq (\mu||X-Y||^p)^{1/p}$.

So $(\mu||X+Y||^p)^{1/p} \leq 2(\mu||X-Y||^p)^{1/p}$.

ie $\left(\int ||\varepsilon_1 X_1 + \ldots + \varepsilon_n X_n||^p dP\right)^{1/p} \leq 2\left(\int ||X_1 + \ldots + X_n||^p dP\right)^{1/p}$.

DEFINITION 1.1. A sequence $(x_n)$ in a Banach space is said to be a basic sequence if $(x_n)$ is a Schauder base for its closed linear span $[x_n]$. A basic sequence $(x_n)$ is called unconditional if, whenever the series $\Sigma a_n x_n$ converges, it converges unconditionally. The following characterizations of basic and unconditional basic sequences are well known and can be found in [3].

PROPOSITION 1.1. (i) A sequence $(x_n)$ is basic if and only if there exists a number $k > 0$ such that for all positive integers m and n with $m \leq n$, and all scalars $a_1, \ldots, a_n$ one has $||\sum_{j=1}^m a_j x_j|| \leq k||\sum_{j=1}^n a_j x_j||$. (ii) A basic sequence $(x_n)$ is unconditional if and only if for all sequences of signs $(\varepsilon_n)$, $\Sigma \varepsilon_n a_n x_n$ converges whenever $(a_n)$ is a sequence of scalars such that $\Sigma a_n x_n$ is convergent.

LEMMA 1.3. If $\{X_n\}$ is a sequence of independent, mean zero random variables in $L_E^p(\Omega)$ $(1 \leq p < \infty)$ then $\{X_n\}$ is an unconditional basic sequence.

PROOF. Let $\{\alpha_n\}$ be any sequence of scalars. Then $\{\alpha_n X_n\}$ is independent, mean zero. So by Lemma 1.1 $(\mu||\sum_{j=1}^m \alpha_j X_j||^p)^{1/p} \leq (\mu||\sum_{j=1}^n \alpha_j X_j||^p)^{1/p}$ if $m \leq n$. This shows that $\{X_n\}$ is basic. Furthermore, if we assume that $\sum_{n=1}^\infty \alpha_n X_n$ is convergent (in $L_E^p(\Omega)$) then for any choice of signs $\{\varepsilon_n\}$ and $m < n$, Lemma 1.2 gives

$$(\mu||\sum_{j=m}^n \varepsilon_j \alpha_j X_j||^p)^{1/p} \leq 2^{1/p}(\mu||\sum_{j=m}^n \alpha_j X_j||^p)^{1/p}.$$

Therefore, $\sum_{n=1}^{\infty} \epsilon_n \alpha_n X_n$ converges in $L_E^p(\Omega)$. Consequently $\{X_n\}$ is unconditional. We shall now show that in $L_E^p(\Omega)$ every sequence of independent mean zero random variables spans a subspace of infinite codimension. In [3] there is an elementary proof for the case $p = 2$ and $E = R$. Our result is almost immediate in view of the following fact whose proof is in [4].

LEMMA 1.4. If F is any Banach space which has an unconditional basis then F does not contain an isomorphic copy of $L^1[0,1]$.

THEOREM 1.1. Assume $(\Omega, \quad , P)$ is probability space such that $L^1(\Omega)$ contains an isomorphic copy of $L^1[0,1]$. If $1 \leq p < \infty$, E is a Banach space and if $\{X_n\}$ $L_E^p(\Omega)$ is an independent mean zero sequence then $[X_n]$, the closed linear span of $\{X_n\}$ in $L_E^p(\Omega)$ has infinite codimension.

PROOF. It is easily seen that if $L^1(\Omega)$ contains an isomorphic copy of $L^1[0,1]$ then so does $L_E^1(\Omega)$. Since $\Omega$ is a probability space $L_E^p(\Omega)$ is a dense subspace of $L_E^1(\Omega)$ for
$1 \leq p < \infty$. Therefore, if $[X_n]$ has finite codimension in $L_E^p(\Omega)$ then it has finite codimension in $L_E^1(\Omega)$. Thus it suffices to establish the assertion for $L_E^1(\Omega)$. Suppose that for some independent sequence $\{X_n\}$ $L_E^1(\Omega)$, $[X_n]$ is of finite codimension m(say). Let $\{Y_1,\ldots,Y_m\}$ be a base for the subspace complementary to $[X_n]$. Then $\{Y_1,\ldots,Y_m,X_1,\ldots,X_n\ldots\}$ is a Schauder base for $L_E^1(\Omega)$. From the results in section 1, $\{X_n\}$ is an unconditional basic sequence. Therefore, the above described base is an unconditional base for $L_E^1(\Omega)$. This is not possible since $L_E^1(\Omega)$ contains an isomorphic copy of $L^1[0,1]$.

2. The notion of finite representability as well as the notions of finite and infinite tree properties were introduced by R.C. James, ([5], [6]) who also characterized super reflexivity in terms of infinite trees. We shall give the definitions and theorems used to obtain our results.

DEFINITION 2.1. A Banach space F is said to be finitely representable in the Banach space E if the following condition holds. For every $\varepsilon > 0$ and any finite dimensional subspace $F_0$ of F there is an into isomorphism $T:F_0 \to E$ such that
$(1 - \varepsilon)||x||_F \leq ||Tx||_E \leq (1 + \varepsilon)||x||_F$ for all $x \in F_0$.

DEFINITION 2.2. A Banach space E is said to be super-reflexive if every Banach space finitely representable in E is reflexive.

DEFINITION 2.3. Let $0 < \delta \leq 2$ and let n be a positive integer. An $(n,\delta)$ tree (in a Banach space) is a finite sequence $\{x_1,x_2,\ldots,x_{2^n+1}\}$ such that $x_i = \frac{x_{2i} + x_{2i+1}}{2}$ for admissible i, and $||x_i - x_{2i+1}|| \geq \delta$, $||x_i - x_{2i-1}|| \geq \delta$.

The following theorem is due to R.C. James.

THEOREM 2.1. A Banach space E is super-reflexive if and only if for each $\delta > 0$ there exists $n \in N$ such that the unit ball of E does not contain an $(n,\delta)$ tree.

We also utilize the following definitions and theorems.

DEFINITION 2.4. A Banach space E is said to be B-convex if $l_1$ is not finitely representable in E.

THEOREM 2.2. Let E be a Banach space with unconditional base. Then the following are equivalent:

(i) E is B-convex

(ii) E is reflexive

(iii) E is super-reflexive

A proof of this theorem appears in the lecture notes of Woyczynski [7]. It is known that the property of super-reflexivity is stronger than the property of B-convexity. Also, B-convexity does not imply and is not implied by reflexivity.

We now state and prove our result.

THEOREM 2.3. Assume that E is either a reflexive or B-convex Banach space. Let $1 < p < \infty$, and $\{f_\lambda\}_{\lambda \varepsilon L}$ be a family of independent, mean zero random variables in $L^p_E(\Omega)$. Then its closed linear span, $[f_\lambda]_{\lambda \varepsilon L}$ is super-reflexive.

PROOF. It is known that if E is B-convex (respectively reflexive) then for $1 < p < \infty$, $L^p_E(\Omega)$ is also B-convex (respectively reflexive). Further closed subspaces of B-convex (reflexive) spaces are B-convex (reflexive). Suppose that $[f_\lambda]_{\lambda \varepsilon L}$ is not super-reflexive. Then by the negation of theorem 2.1, there is $\delta > 0$ such that for each n there is an $(n,\delta)$ tree contained in the unit ball of $[f_\lambda]_{\lambda \varepsilon L}$. Let G be the closed linear span of the union of these $(n,\delta)$ trees. Then G is separable since the above union is countable. We claim that there is a countable set $\{f_n\}_n$ $\{f_\lambda\}_{\lambda \varepsilon L}$ such that G $[f_n]_n$.

Indeed, since G is separable, we may choose a sequence $\{Y_n\}_{n \varepsilon N}$ G which is dense in G. For each $Y_n$, there is sequence $\{Z_k^{(n)}\}_k$ of finite linear combinations of the $f_\lambda$ such that $Z_k^{(n)} \to Y_n$ as $k \to \infty$. Thus for each $Y_n$, there is a countable subfamily $\{f_k^{(n)}\}_k \subseteq \{f_\lambda\}_{\lambda \epsilon L}$ such that $Y_n \epsilon [f_k^{(n)}]_k$. Now $\bigcup_{n=1}^{\infty} \bigcup_k \{f_k^{(n)}\}_{n,k}$ is a countable subfamily of $\{f_\lambda\}_{\lambda \epsilon L}$ and $G \subseteq [f_k^{(n)}]_{n,k}$. By the results of Section 1, $\{f_k^{(n)}\}_{n,k}$ is an unconditional basic sequence. Therefore, the subspace $[f_k^{(n)}]_{n,k}$ has unconditional basis. Since this subspace is B-convex (reflexive) it is, in view of Theorem 2.2 also super-reflexive. But the unit ball of $[f_k^{(n)}]_{k,n}$ contains the unit ball of $G$ which in turn contains $(n,\delta)$ tress for all $n$.

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