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MACKEY CONVERGENCE AND QUASI-SEQUENTIALLY WEBBED SPACES

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ABSTRACT : The problem of characterizing those locally convex spaces satisfying the Mackey convergence condition is still open. Recently in [4], a partial description was given using compatible webs. In this paper, those results are extended by using quasi-sequentially webbed spaces (see Definition 1). In particular, it is shown that strictly barrelled spaces satisfy the Mackey convergence condition and that they are properly contained in the set of quasi-sequentially webbed spaces. A related problem is that of characterizing those locally convex spaces satisfying the so-called fast convergence condition. A partial solution to this problem is obtained. Several examples are given.

KEY WORDS AND PHRASES: Webbed space, quasi-sequentially webbed space, Mackey convergence, fast convergence, local completeness, inductive limit.

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1. Introduction and Definitions.

In [7], 28.3, Köthe pointed out that a characterization of locally convex spaces satisfying the Mackey convergence condition (see definition below) did not exist. This problem is still open. In [4], a partial solution is given, using spaces with web structures. In this paper, those results are extended. Also, the related problem of describing those locally convex spaces satisfying the fast convergence condition (defined below) is briefly examined. Several examples are given along the way.

Throughout this paper, E will denote a Hausdorff locally convex space. If A is a subset of E which is absolutely convex, we will call A a disk, and we let E_A denote the linear span of A, endowed with the topology generated by the Minkowski functional of A. When A is bounded, E_A is a normed space, and the normed topology is finer than the topology inherited from E. If E_A is a Banach space, we call A a Banach disk. A locally convex space is locally complete if each closed, bounded disk is a Banach disk. If (x_n) is a sequence in E which converges to x in the normed space E_A for some bounded disk A, we say that (x_n) is Mackey (or locally) convergent to x. In case x = 0, we say (x_n) is Mackey null. Because all topologies involved here are translation invariant, we will always use null sequences. Also, it is obvious that every locally null sequence is a null sequence for the original topology on E. When each null sequence is locally null, we say that E satisfies the Mackey convergence condition. If (x_n) is a Mackey null sequence and the corresponding disk A is a Banach disk, (x_n) is fast convergent to 0. If every null sequence is fast convergent to 0, then E satisfies the fast convergence condition.

We will also need the following information on webs. Detailed discussions of webs may be found in [3], [10], and [11]. A web on a locally convex space will be denoted by $\mathcal{W}$. A sequence $\{W_{m_1, m_2, \ldots m_k}: k \in N\}$ of members from a web $\mathcal{W}$ such that $W_{m_1, \ldots, m_{k+1}} \subset W_{m_1, \ldots, m_k}$ is called a strand. For convenience, we will denote a strand by (W_k).
Also, we assume throughout this paper, that for a strand (W_k) of $\mathcal{W}$,

$$W_{k+1} \subset \frac{1}{2} W_k.$$

If $\mathcal{W}$ is a web on a locally convex space E, we say that $\mathcal{W}$ is compatible with E if given any zero neighborhood U in E, and any strand (W_k) from $\mathcal{W}$, there exists $k \in N$ such that $W_k \subset U$. A compatible web $\mathcal{W}$ is completing if for each strand (W_k) from $\mathcal{W}$ and for each series $\sum_{k=1}^{\infty} x_k$, with $x_k \in W_k$ for each $k \in N$, $\sum_{k=1}^{\infty} x_k$ is convergent in E. In this case we say E is webbed. If $\mathcal{W}$ is completing and for each strand (W_k) and each series $\sum_{k=1}^{\infty} x_k$ with $x_k \in W_k$, we have

$$\sum_{r=k+1}^{\infty} x_r \in W_{k-1},$$

then $\mathcal{W}$ is strict, and we say that E is strictly webbed.

2. Quasi-Sequentially Webbed Spaces and Mackey Convergence

One obvious property of compatible webs is that the members of any strand can be made small enough to fit into any zero neighborhood. If each null sequence has the property that its members are eventually contained in a finite collection of strands, then it appears that the sequence is converging with respect to a finer topology. This is the idea behind sequentially webbed spaces and quasi-sequentially webbed spaces.

Definition 1: Let E be a locally convex space with a compatible web $\mathcal{W}$. Then E is sequentially webbed if for each null sequence (x_n) there exists a finite collection of strands $\{(W_k^{(1)}), (W_k^{(2)}), \ldots, (W_k^{(m)})\}$ from $\mathcal{W}$ such that for each $k \in N$ there exists $N_k \in N$ such that

$$x_n \in \bigcup_{i=1}^{m} W_k^{(i)},$$

for each $n \geq N_k$. If we have the weaker condition, that for each $n \geq N_k$

$$x_n \varepsilon \bigcup_{i=1}^{m} V_k(i),$$

where V_k is the closed, convex balanced hull of W_k, then we say E is quasi-sequentially webbed.

Remark: Every sequentially webbed space is quasi-sequentially webbed. An example of a quasi-sequentially webbed space which is not sequentially webbed is given below.

Example 1: Let $E = \ell^{1/2}$, with the non-locally convex metrizable topology ζ generated by the decreasing sequence $\{W_n: n \varepsilon N\}$. It is easy to show that $\mathcal{W} = \{W_n: n \varepsilon N\}$ is a compatible web for (E,ζ). Let η denote the topology on E induced by the normed topology on ℓ^1. $\mathcal{W}$ is then a compatible web for η also. Now let $\mathcal{W}' = \{W'_n: n \varepsilon N\}$, where W'_n is the ℓ^1- closure of the convex balanced hull of W_n, for each $n \varepsilon N$. Since $\mathcal{W}'$ forms a base of closed zero neighborhoods for η , (E,η) is quasi-sequentially webbed under $\mathcal{W}'$. On the other hand, we may pick $x_n \varepsilon W'_n \backslash W_n$ for each n, so that $x_n \rightarrow 0$, but is not contained in the (only) strand (W_n). Hence, E with the web $\mathcal{W}$ is not sequentially webbed.

In the following proposition, let $\{E_n: n \varepsilon N\}$ be a collection of locally convex spaces with $E_n \subset E_{n+1}$ for each n, and each injection id: $E_n \rightarrow E_{n+1}$ continuous. Then we write $E = \text{ind}_n \lim E_n$ to represent the inductive limit of the spaces E_n. An inductve limit $E = \text{ind}_n \lim E_n$ is sequentially retractive if for each convergent sequence in E there is some $n \varepsilon N$ such that the sequence converges to the same limit in E_n.

Theorem 1:

(a) Every metrizable locally convex space is quasi-sequentially webbed.

(b) Let $E = \text{ind}_n \lim E_n$ be sequentially retractive, with each E_n closed in E. If each E_n is quasi-sequentially webbed, then so is E.

(c) Every strict (LF)-space is quasi-sequentially webbed.

(d) Every subspace of a quasi-sequentially webbed space is quasi-sequentially webbed.

Proof:

(a) Let $\mathcal{W} = \{U_n: n \varepsilon N\}$ be a base of zero neighborhoods of the metrizable space E consisting of absolutely convex, closed sets U_n, with $U_{n+1} \subset 1/2\, U_n$. Clearly, $\mathcal{W}$ is a compatible web for E and E is quasi-sequentially webbed under $\mathcal{W}$.

(b) Let $E = \text{ind}_n \lim E_n$ satisfying the hypothesis. For each n, let $\mathcal{W}^{(n)}$ denote the web on E_n. Define the web

$\mathcal{W} = \{W_{m_1,\ldots,m_k} : k, m_1, \ldots, m_k \varepsilon N\}$ on E as follows:

Let $\{W_{m_1}: m_1 \varepsilon N\}$ consist of the collection $\{W^{(n)}_{\ell_1} : \ell_1, n \varepsilon N\}$ where the subscripts are put in one-one correspondence.

Let $\{W_{m_1,m_2}: m_1,m_2 \in N\}$ be the collection $\{W^{(n)}_{\ell_1,\ell_2}: \ell_1,\ell_2, n \in N\}$, and so on. It is easy to verify that $\mathcal{W}$ is a web on E, and $\mathcal{W}$ is compatible with E by [4], Proposition 9.

Now let $x_m \to o$ in E. Then $x_m \to o$ in E_n for some $n \in N$. Hence, there are ℓ strands

$(W_k^{(n,1)}), \dots (W_k^{(n,\ell)})$

in $\mathcal{W}^{(n)}$ such that for each $k \in N$, there is $N_k \in N$ such that

$$x_m \in \bigcup_{i=1}^{\ell} V_k^{(n,i)}$$

for each $m \geq N_k$, where $V_k^{(n,i)}$ is the E_n-closure of $W_k^{(n,i)}$, for $i = 1, \dots, \ell$ and $k \in N$. Each $V_k^{(n,i)}$ is closed in E_n which in turn is closed in E; hence, each $V_k^{(n,i)}$ is closed in E.

Moreover, by the construction of $\mathcal{W}$ on E, each strand of $\mathcal{W}^{(n)}$ is a strand of $\mathcal{W}$. Hence, E is quasi-sequentially webbed.

(c) Every strict (LF)-space satisfied the assumption in (b).

(d) Let E be a quasi-sequentially webbed space and let F be a subspace of E. If $\mathcal{W}$ is the web on E, define the web $\mathcal{W}'$ on F by

$\mathcal{W}' = \{W_{m_1,\dots,m_k} \cap F : k, m_1 \dots, m_k \in N\}$

It is routine to show that $\mathcal{W}'$ is a compatible web on F. Next, if $x_n \to o$ in F,$x_n \to o$ in E, so there are strands

$(W_K^{(1)}), \dots, (W_K^{(m)}) \subset \mathcal{W}$

such that for each $k \in N$, there is $N_k \in N$ so that

$$x_n \in \bigcup_{i=1}^{m} V_k^{(i)},$$

where V_k is the E-closure of the convex, balanced hull of $W^{(i)}_k$

Hence,

$$\begin{aligned} x_n &\in \left(\bigcup_{i=1}^{m} V_k^{(i)}\right) \cap F \\ &= \bigcup_{i=1}^{m} (V_k^{(i)} \cap F) \\ &= \bigcup_{i=1}^{m} V_k^{(i)'}, \end{aligned}$$

where $V_k^{(i)'}$ is the F-closure of the convex, balanced hull of $W_k^{(i)} \cap F$.

This shows that F is quasi-sequentially webbed.

Remark: If all the spaces E_n in (b) are in fact sequentially webbed, then the assumption that each E_n is closed in E may be dropped, since in this case, E is sequentially webbed by [4], Proposition 9.

The proof of our next result is similar to the one for [4], Theorem 12; its proof is left to the reader.

Theorem 2: Every quasi-sequentially webbed space satisfies the Mackey convergence condition.

Remark: The motivation for defining quasi-sequentially webbed spaces is in the two corollaries below; they allow us to enlarge the list of locally convex spaces which satisfy the Mackey convergence condition. First, we need the following definition, given recently by Valdivia [12] in connection with the closed graph theorem. In this context, a web is ordered if given arbitrary positive integers k, $m_1, \ldots, m_k$ and $n_1, \ldots n_k$ such that $m_i \leq n_i$; for i = 1, ..., k, then

$$W_{m_1,\ldots,m_k} \subset W_{n_1,\ldots,n_k}$$

Definition 2: A locally convex space E is strictly barrelled if given any ordered and absolutely convex web $\mathcal{W} = \{W_{m_1,\ldots,m_k} : k, m_1,\ldots,m_k \in N\}$ on E, there is a sequence (m_n) of positive integers such that $\overline{W}_{m_1,\ldots,m_n}$ is a zero neighborhood in E, for each $n \in N$.

Remark: Strictly barrelled spaces are studied in detail in Section 6 of [12]. We note here that strictly barrelled spaces include unordered Baire-like spaces properly. (See [12] again).

Corollary 1: Every strictly barrelled locally convex space satisfies the Mackey convergence condition.

Proof: Let E be strictly barrelled and let $\mathcal{W}$ by any ordered and absolutely convex web on E. Define the web $\mathcal{W}'$ by

$$\mathcal{W}' = \{2^{-k} W_{m_1, m_2, \ldots, m_k} : k, m_1, m_2, \ldots, m_k \in N\}$$

Then $\mathcal{W}'$ is another ordered, absolutely convex web on E, which satisfied the condition

$$W_{k+1} \subset \frac{1}{2} W_k$$

for each strand (W_k) of $\mathcal{W}'$. Because E is strictly barelled, there is a strand (W_k) of $\mathcal{W}'$ such that $\overline{W}_k$ is a zero neighborhood of E for each $k \in N$. Thus, E is quasi-sequentially webbed.

Corollary 2: If E is a Baire space with a compatible web, then E satisfies the Mackey convergence condition.

Proof: By lemma 2 page 158 of [11], if E is a Baire space with a compatible web $\mathcal{W}$ then there is a strand (W_k) of $\mathcal{W}$ such that $\overline{W}_k$ is a zero neighborhood in E for each $k \in N$. This makes E quasi-sequentially webbed.

We will now obtain a partial converse to Theorem 2. We note that in [4], Theorem 18 it is shown that if E is locally complete, strictly webbed, and satisfies the Mackey convergence condition, then E is sequentially webbed. We will generalize this. First, we need to introduce the following:

Definition 3: A locally convex space E is locally Baire if for each bounded subset A of E there exists a bounded disk $B \subset A$ such that E_B is a Baire space.

Remark: Every locally complete space is locally Baire. Moreover, in [2], page 3-4, example 6, an example of a normed Baire space which is not complete is given; this represents a locally Baire space which is not locally complete. Also, any strict (LF)-space represents an example of a locally Baire space which is neither Baire nor metrizable.

Theorem 3: Let E be webbed and locally Baire. If E satisfies the Mackey convergence condition then E is quasi-sequentially webbed.

Proof: Let $x_n \to o$ in E. Using Kothe [7], 28.3, there is a sequence $(r_n) \subset (o,\infty)$ such that $r_n \to \infty$ as $n \to \infty$, and $r_n x_n \to o$ in E. Let $A = \{r_n x_n : n \in N\}$. Then A is bounded so there exists a bounded disk $B \supset A$ such that E_B is a Baire space. The injection $id: E_B \to E$ is continuous so it has a closed graph. If E is webbed using $\mathcal{W}$, then using Theorem 19, cor. 1 page 722 of [10], there is a strand $(W_k) \subset \mathcal{W}$ and there is a sequence (α_k) of numbers such that

$$id\, B = B \subset \alpha_k \overline{W}_k$$

for each $k \in N$. Hence, for each $n \in N$

$$r_n x_n \in \alpha_k \overline{W}_k \subset \alpha_k V_k$$

where $V_k = \text{convbal}(\overline{W}_k) = \overline{\text{convbal}(W_k)}$. Thus, for each fixed $k \in N$, we find $N_k \in N$ so that $\frac{|\alpha_k|}{r_n} < 1$ for each $n \geq N_k$.

Then we have

$$x_n \in \frac{|\alpha_k|}{r_n} V_k \subset V_k$$

since each V_k is balanced.

Corollary: Let E be locally Baire and webbed. Then E satisfies the Mackey convergence condition if and only if E is quasi-sequentially webbed.

Example 2: We may use the above result to find an example of a quasi-sequentially webbed space which is not strictly barrelled. Let E be the strong dual of a Fréchet-Schwartz space. Then E is complete, hence locally Baire. Moreover, E is webbed by Proposition 2 page 157 of [11]. In fact the web $\mathcal{W}$ on E is described as follows:

Let $\{U_n : n \in N\}$ be a decreasing base of zero neighborhoods for the Fréchet-Schwartz space F such that E is the strong dual of F. We define $\{W_{m_1} : m_1, \in N\} = \{U_n^o : n \in N\}$, where U_n^o is the polar of U_n for each $n \in N$. Then define

$$\{W_{m_1, m_2} : m_1, m_2 \in N\} = \{\tfrac{1}{2} U_n^o : n \in N\},$$

and, in general,

$$\{W_{m_1, \dots, m_k} : k, m_1, \dots, m_k \in N\} = \{\tfrac{1}{2^{k-1}} U_n^o : n \in N\},$$

Clearly, $\mathcal{W} = \{W_{m_1,\ldots,m_k} : k, m_1,\ldots, m_k \in N\}$ is an ordered web on E consisting of absolutely convex, closed sets. Furthermore, E is not normable, so none of the members of $\mathcal{W}$ can be zero neighborhoods in E, which means that E cannot be strictly barrelled. Finally, by 12.5.9 of [5], E satisfies the Mackey convergence condition, so by Theorem 3, E is quasi-sequentially webbed.

3. The Fast Convergence Condition.

In [6], a sequence is defined to be fast convergent in a locally convex space E if there is a compact disk B in E such that (x_n) is convergent in E_B. This differs from our definition, but this difference is easily rectified in theorem 4 below. Also, in [6], it is shown that a locally complete bornological space satisfies the Mackey convergence condition if and only if it satisfies the fast convergence condition. In this section we will show that "bornological" may be removed from that statement and that the only difference between a locally convex space satisfying the Mackey convergence condition and one satisfying the fast convergence condition is the presence of local completeness. Finally, we will make some statements connecting this section with the previous section.

Theorem 4: Let E be a locally convex space. Then the following are equivalent:

(a) E satisfies the fast convergence condition.

(b) For each null sequence (x_n) in E there is a compact disk K in E such that $x_n \to o$ in E_K.

(c) E is locally complete and satisfies the Mackey convergence condition.

Proof: To show (a)$\Rightarrow$(b), let $x_n \to o$ in E_B where B is a bounded Banach disk. Then since E_B is a Fréchet space, 35.7,(4) of [8] applies; namely, there is a compact disk K such that $x_n \to o$ in E_K. Next, to show that (b)$\Rightarrow$(c) note that under the assumptions in (b), we need only show that E is locally complete. To do this, we use 5.1.11 of [9], where it is shown that a locally convex space is locally complete if the closed absolutely convex hull of each null sequence is compact. Let (x_n) be a null sequence in E. Then $x_n \to o$ in E_K for some compact disk K in E. E_K is a Banach space ([9], 3.2.5) so if A denotes the E_K - closure of convbal $(\{x_n : n \in N\})$, then A is compact in E_K. Since the injection id: $E_K \to E$ is continuous, A is compact in E, too. The assertion is now obtained by showing that the E -closure of convbal $(\{x_n : n \in N\})$ is A. Denote by A_o, the closure of A in E. Since A is bounded in E_K, there is a $\lambda > o$ such that $A \subset \lambda K$. λK is closed in E, so we have $A_o \subset \lambda K \subset E_K$. Now take $x_o \in A_o$ and let (x_α) be a net in A such that $x_\alpha \to x_o$ in the topology of E. Note that id: $E_K \to E$ is continuous and that $\{n^{-1} K : n \in N\}$ is a base of zero neighborhoods for E_K consisting of sets closed in E, hence also closed in the linear hull of K. Thus, by 3.2.4 of [5], $x_\alpha \to x_o$ in the topology of E_K. Moreoever, A is E_K-closed, so $x_o \in A$, which shows that $A_o = A$. Finally, (c)$\Rightarrow$(a) is obvious.

Corollary 1: Any locally convex space satisfying the fast convergence condition is locally complete.

Corollary 2: Any metrizable, incomplete locally convex space satisfies the Mackey convergence condition but not the fast convergence condition.

Proof: Any such space is quasi-sequentially webbed by Theorem 1(a), but cannot be locally complete by [1], II.2.

Corollary 3: If E is locally complete, then E satisfies the fast convergence condition if and only if E satisfies the Mackey convergence condition.

The next two results are combinations of Theorem 4 above and results from the previous section. They give characterizations of locally convex spaces satisfying the fast convergence condition for the case where the spaces are webbed.

Theorem 5: Let E be a webbed locally convex space. Then the following are equivalent:

(a) E satisfies the fast convergence condition.

(b) E is locally complete and quasi-sequentially webbed.

Proof: This is an immediate consequence of Theorems 2, 3, and 4.

For the result below, we recall that an inductive limit E of locally convex spaces is regular if each bounded set in E is contained in and bounded in one of the constituent spaces.

Theorem 6: Let E be a regular inductive limit of locally complete webbed spaces. Then the following are equivalent:

(a) E satisfies the fast convergence condition.

(b) E satisfies the Mackey convergence condition.

(c) E is quasi-sequentially webbed.

Proof: To show (b) <=> (a), it suffices by Corollary 3 of Theorem 4 to show that E is locally complete. Hence, let A be a bounded subset of E, where $E = \text{ind}_n \lim E_n$. Then A is bounded in E_{n_0} for some $n_0 \in N$. Thus, if B is the E_{n_0}-closure of convbal (A), then B is a bounded Banach disk in E_{n_0}, hence also in E. Moreoever, $A \subset B$, so we have shown that every bounded subset of E is contained in a Banach disk. It follows now by [9], 5.1.6, that E is locally complete. Finally, we obtain (a) <=> (c) by noting that an inductive limit of webbed spaces is webbed ([3], IV.4.6); the assertion then follows from the local completeness of E and Theorem 5.

As in the previous section, we give some examples.

Example 3: It is shown in [4], Example 15, that ℓ^1 with its weak topology is a locally complete, non-bornological, non-metrizable locally convex space satisfying the Mackey convergence condition. This space also satisfies the fast convergence condition since it is locally complete.

<u>Example 4:</u> In this example, we show that there are locally complete webbed spaces which do not satisfy the Mackey convergence condition, hence, not the fast convergence condition either. Let (E, ||.||) be any Banach space having weakly convergent sequences that are not norm convergent. For instance, E could be the Banach space $L^p([0,1])$, where $1 \leq p \leq \infty$. Let B denote the closed unit ball of E. Then the web $\mathcal{W} = \{2^{-n}B : n \varepsilon N\}$ is a compatible web on E for which E is webbed. Moreoever, we show that E with its weak topology σ is also webbed with respect to $\mathcal{W}$.

First, $\mathcal{W}$ is compatible with σ since each weak zero neighborhood contains some member of $\mathcal{W}$. Next, let (x_n) be any sequence in E such that $x_n \varepsilon 2^{-n} B$ for each $n \varepsilon N$. Since $\mathcal{W}$ is a completing web with respect to ||.||, $\sum_{n=1}^{\infty} x_n$ is norm convergent in E, hence this series is also weakly convergent in E. Thus, $\mathcal{W}$ is completing for σ.

Furtheremore, it is clear that a closed, bounded disk in E is a Banach disk with respect to T if and only if it is a Banach disk with respect to T^1, where T and T^1 are any topologies which are compatible with respect to the duality $\langle E,E' \rangle$. Hence, since (E, ||.||) is complete, (E, σ) is locally complete. Therefore, by corollary 1 of Theorem 3, (E,σ) satisfies the Mackey convergence condition if and only if (E, σ) is quasi-sequentially webbed. However, if (x_n) is a weakly null sequence which is not norm convergent, then (x_n) cannot be contained in the (only) strand $(2^{-n} B)$ of $\mathcal{W}$, since this would imply the norm convergence of (x_n). Thus, (E,σ) is not quasi-sequentially

REFERENCES

1. BOSCH, C., KUCERA, J., and MCKENNON, K., Fast Complete Locally Convex Linear Topological Spaces, Internat. J. Math and Math. Sci. 9, no. 4, 1986, (791-796).

2. BOURBAKI, N., Elements de Mathematique, Livre V (Espaces Vectoriels Topologiques), Chapt. III, Hermann, 1955.

3. DEWILDE, M., Closed Graph Theorems and Webbed Spaces, Pitman, 1978.

4. GILSDORF, T., The Mackey Convergence Condition for Spaces with Webs, Internat. J. Math. and Math. Sci., 11, no. 3, 1988, (473-484).

5. JARCHOW, H., Locally Convex Spaces, B.G. Teubner Stuttgart, 1981.

6. JARCHOW, H. and SWART, J., On Mackey Convergency in Locally Convex Spaces, Israel J. Math., 16, 1973, (150-158).

7. KÖTHE, G., Topological Vector Spaces I, Springer, 1969.

8. KÖTHE, G., Topological Vector Spaces II, Springer, 1979.

9. PÉREZ CARRERAS, P., and BONET, J., Barrelled Locally Convex Spaces, North Holland Mathematics Studies, No. 131, 1987.

10. ROBERTSON, W., On the Closed Graph Theorem and Spaces with Webs, Proc. London Math. Soc., 24, 1972, (692-738).

11. ROBERTSON, A.P., and ROBERTSON, W., Topological Vector Spaces, Second Edition, Cambridge University Press, 1973.

12. VALDIVIA, M., Quasi-LB Spaces, J. London Math. Sco. (2) 35, 1987, (149-168).

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