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A REMARK ON THE WEIGHTED AVERAGES FOR SUPERADDITIVE PROCESSES

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ABSTRACT. A decompostion of a superadditive process into a difference of an additive and a positive purely superadditive process is obtained. This result is used to prove an ergodic theorem for weighted averages of superadditive processes.

KEY WORDS AND PHRASES. Ergodic theorem, bounded sequences, superadditive process, additive process.
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1. INTRODUCTION. Let (X, μ) be a probability space and let $L_p(X, \ ,\mu)$, $1\leq p\leq\infty$ be the classical Banach space of real valued functions f with $\int |f|^p d\mu = ||f||_p^p < \infty$. Let $T:L_p \to L_p$ be a linear operator. A family of L_p functions $F = \{F_n\}_{n\geq 1}$ is called a T-superadditive process if

$$F_{n+m} \geq F_n + T^n F_m, \text{ for all } n, m \geq 1, \tag{1.1}$$

and F is called a T-additive process if equality holds in (1.1). Notice that if we let $f_i = F_{i+1}-F_i$, for $i\geq 0$ we have $F_n = \sum_{i=0}^{n-1} f_i$, where $F_o \equiv 0$, for all $n \geq 1$. Consider a sequence $A = \{a_n\}_{n\geq 0}$ of complex numbers and a T-(super) additive process F. We define a family of L_p-functions $(F,A)=\{a_n f_n\}_{n\geq 0}$, and set $S_n(F,A) = \sum_{i=0}^{n-1} a_n f_n$. If A is the constant sequence $1=(1,1,\ldots.)$, then $S_n(F,A) = F_n$.

In the following, we observe that the weighted and subsequential ergodic theorems for T-superadditive processes are direct consequences of their T-additive counterparts.

2. THE DECOMPOSITION OF F.

In this section a decomposition of a T-superadditive process F into a difference of a T-additive process G and a positive, purely T-superadditive process H (that is, H is a positive T-superadditive process that does not dominate any non-zero positive T-additive process) is obtained.

CASE p=1. Let T be a positive Dunford-Schwartz operator i.e., T is an L_1-contraction with $||T||_\infty \leq 1$. We will also assume that T is Markovian, that is $\int Tf\,d\mu = \int f\,d\mu$. In this case, if $\sup_{n\geq 1} \frac{1}{n} ||F_n||_1 < \infty$, then the decomposition result is obtained by M.A. Akcoglu and L. Sucheston [1]. Namely, they obtained that for all $n \geq 1$,

$$F_n = G_n - H_n \tag{2.1}$$

where $G_n = \sum_{i=0}^{n-1} T^i\delta$ for some $\delta\varepsilon L_1$, and $H_n = \sum_{i=0}^{n-1} h_i$ with $h_i = f_i - T^i\delta$. Using this result they showed that $\lim_{n\to\infty} \frac{1}{n} F_n$ exists a.e., and moreover it is a consequence of the same result that $\lim_{n\to\infty} \frac{1}{n} H_n = 0$ a.e.

CASE $1 < p < \infty$. In this case we let T be a positive L_p-contraction and F a T-superadditive process with $\lim_{n\to\infty} \inf ||\frac{1}{n} \sum_{i=0}^{n} (F_i - TF_{i-1})||_p < \infty$. Under these conditions B. Hachem [2] showed that $\lim_{n\to\infty} \frac{1}{n} F_n$ exists a.e. by reducing the problem to a problem in an appropriate L_1-space and employing Akcoglu-Sucheston's result in case p=1 above. Here we observe that the same technique can be applied to yield to a decomposition result similar to (2.1).

Using a result of M.A. Akcoglu and L. Sucheston [3] one can decompose X uniquely into disjoint union of sets E and E^c where:

(i) E is the support of a T-invariant function $h\varepsilon L_p^+$, and supp $g \subset E$ for all T-invariant $g\varepsilon L_p^+$.

(ii) $L_p(E)$ and $L_p(E^c)$ are invariant subspaces for T.

Then the following results are obtained [2,3]:

$$1_{E^c}\left(\frac{F_n}{n}\right) \to 0 \text{ a.e., so one can assume that } X=E. \tag{2.2}$$

The operator $P:L_p(m) \to L_p(m)$ defined as $Pf = \frac{T(fh)}{h}$, $f\varepsilon L_p(m)$, is a positive $L_p(m)$-contraction and $P1=1$, where $m=h^p\cdot\mu$. (2.3)

In particular, $\int Pfdm = \int fdm$ for all $f\varepsilon L_p^+(m)$. So P can be extended to a Markovian operator on $L_1(m)$. Consequently $F'=\{h^{-1}F_n\}$ is a bounded P-superadditive process [2] in $L_1(m)$. Now by applying the Akcoglu-Sucheston's result [1] we can decompose F' into a difference of a P-additive process and a positive, purely P-superadditive process as $h^{-1}F_n=G_n-H_n$, $n \geq 1$. Also we see that:

$$\lim_{n\to\infty} \frac{1}{n}(h^{-1}F_n) \text{ exists m-a.e. and } \lim_{n\to\infty}\frac{1}{n}H_n = 0 \text{ m-a.e.,} \tag{2.4}$$

so $F_n = hG_n-hH_n$, and that $G=\{hG_n\}$ and $H=\{hH_n\}$ are T-additive and T-superadditive processes respectively by (2.3). Consequently (2.4) gives that

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}F_n \text{ exists } \mu\text{-a.e. X and}\\ &\lim_{n\to\infty}\frac{1}{n}H_n = 0 \ \mu\text{-a.e. X}\end{aligned} \tag{2.5}$$

by (2.2) and (2.3)

3. WEIGHTED AVERAGES.

Given a linear operator T on L_p, $1 \leq p \leq \infty$, and a sequence $A=\{a_n\}_{n\geq 0}$ of complex numbers if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} a_k T^k f \text{ exists a.e.}$$

for all $f\varepsilon L_p$, then we say that A is a good weight for T [4], or (A,T) is Birkhoff [5].

R. Sato [6] showed that the uniform sequences are good for $1<p<\infty$. C. Ryll-Nardzewski [7] proved that the bounded Besicovitch sequences are good for T induced by a measure preserving transformation $\phi:X \to X$ by $Tf(x)=f(\phi(x))$ for any $f\varepsilon L_1$. This result combined with the remarkable theorem of J. Baxter and J. Olsen [5, Theorem 2.19] imply that bounded Besicovitch sequences are good for Dunford-Schwartz operators.

Now we observe the following: Let T be an operator on L_p and F be a T-superadditive process. If $F_n=G_n-H_n$, then for any sequence A

$$S_n(F,A) = S_n(G,A) - S_n(H,A) \tag{3.1}$$

Also

Also

$$0 \leq \lim_{n \to \infty} \sup \frac{1}{n} |S_n(A,H)| \leq M \cdot \lim_{n \to \infty} \sup \frac{1}{n} H_n$$

where $M = \sup_{n \geq 1} |a_n|$. Therefore if $\lim_{n \to \infty} \frac{1}{n} H_n = 0$ a.e., $\lim_{n \to \infty} \frac{1}{n} S_n(A,H)=0$ a.e. for any bounded sequence A. We summarize this discussion as

THEOREM 3.2. Let T be a positive Dunford-Schwartz operator on L_p, $1 \leq p < \infty$, and F be a T-superadditive process. Assume also that

(i) T is Markovian and $\sup_{n \geq 1} || \frac{1}{n} F_n ||_1 < \infty$ when p=1,

(ii) $\lim_{n \to \infty} \inf || \frac{1}{n} \sum_{i=1}^{n} (F_i - TF_{i-1}) ||_p < \infty$ $(F_o \equiv 0)$ when $1<p<\infty$.

If A is a bounded sequence such that (A,T) is Birkhoff, then

$$\lim_{n \to \infty} \frac{1}{n} S_n (F,A) \text{ exists a.e.} \tag{3.3}$$

REMARK 3.4. The limit in (3.3) exists a.e. when A is a uniform sequence or a bounded Besicovitch sequence of $A \in \overline{W}_4$ [5]. In particular the subsequence theorem [5, 4] is valid for superadditive processes.

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