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ORTHOGONAL BASES IN A TOPOLOGICAL ALGEBRA ARE SCHAUDER BASES

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ABSTRACT. In a topological algebra with separately continuous multiplication, the result quoted in the title is proved.

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1. INTRODUCTION.

A topological algebra A is a linear associative algebra over complex scalars which is a Hausdorff topological vector space (TVS) in which multiplication is separately continuous, i.e., for each $x \epsilon A$, the operators L_x and R_x, $L_x y = xy$, $R_x y = yx$ $(y \epsilon A)$, are continuous. A basis (e_n) in A is Schauder (respectively b-Schauder) if the functionals e_n^*, $e_n^*(x) = \alpha_n$ (where $x = \sum_1^\infty \alpha_n e_n$), are continuous (respectively bounded i.e. map bounded sets to bounded sets). An orthogonal basis is a basis (e_n) satisfying $e_n\, e_m = \delta_{nm}\, e_n$ for all n, m.

Recently S. El-Helaly and T. Husain [1] showed that an orthogonal basis in A is Schauder if multiplication is jointly continuous (i.e. continuous as a bilinear map on $A \times A$). Now joint continuity is a very stringent requirement. In fact, abundance of examples have forced upon some other weaker modes of continuity in literature. Multiplication in A is hypocontinuous (respectively sequentially jointly continuous) if given a o-neighborhood U and a bounded set B, there is a o-neighborhood V such that $BV \subset U$, $VB \subset U$ (respectively for sequences (x_n), (y_n) in $A, x_n \to x, y_n \to y$ imply $x_n y_n \to xy$). In a topological algebra, joint continuity gives hypocontinuity which in turn implies sequential joint continuity; and if A is barelled (respectively complete matrizable or m-convex), multiplication is hypocontinuous (respectively jointly continuous). We extend the above result of El-Helaly and Husain in its final form by modifying their arguments, and also obtain its variant in a more general frame-work.

2. MAIN RESULTS.

THEOREM. Let A be a Hausdorff TVS that is an algebra

(1) If A is a topological algebra, then every orthogonal basis in A is Schauder.

(2) If multiplication in A is sequentially separately continuous (i.e. for a sequence (x_n) in A, $x_n \to 0$ implies $x_n y \to 0, y x_n \to 0$ for all y), then every orthogonal basis in A is b-Schauder.

PROOF. Let (e_n) be an orthogonal basis in A. Let $n \varepsilon N$ be fixed. Orthogonality applied to the expansion $x = \sum_1^\infty e_n^*(x)e_n$ implies that $e_n x = e_n^*(x)e_n = xe_n$ for all x in A. Choose a balanced o-neighborhood U such that $e_n \not\in U$. Let $r = \inf\{d > 0: e_n \varepsilon dU\}$. Then $r > 1$.

(1) Let (x_α) be a net in A such that $\lim x_\alpha = 0$. Hence $\lim_\alpha x_\alpha e_n = 0$. Given an $\varepsilon > 0$, there is an α_o such that $e_n^*(x_\alpha)e_n = x_\alpha e_n \varepsilon(\varepsilon U)$ for all $\alpha \geq \alpha_o$. As U is balanced, $|e_n^*(x_\alpha)| e_n \varepsilon(\varepsilon U)$ for $\alpha \geq \alpha_o$. Hence by the definition of r, $|e_n^*(x_\alpha)|^{-1}\varepsilon \geq r > 1$, and so $|e_n^*(x_\alpha)| < \varepsilon$ for all $\alpha \geq \alpha_o$. Thus $\lim_\alpha e_n^*(x_\alpha) = 0$.

(2) Since a subset in a TVS is bounded iff each of its countable subset is bounded, it is sufficient to show that e_n^* maps a bounded sequence (x_k) to a bounded sequence. Now for any sequence $r_k \to \infty$, $r_k > 0$, $x_k/r_k \to 0$. By sequential separate continuity of multiplication, $e_n\, x_k/r_k \to 0$. Hence $(e_n^*(x_k)\,(e_n)_{k=1}^\infty$ is bounded, and for all k, $e_n^*(x_k)e_n \varepsilon \lambda U$ for some $\lambda = \lambda(n, U) > 0$. Again by definition of r, $|e_n^*(x_k)| \leq \frac{r}{\lambda}$ for all k.

REMARKS. (1) It follows that Corollaries 1.2 and 2.2 in [1] hold for any topological algebra.

(2) In a topological algebra, a basis which is not orthogonal need not be Schauder even if multiplication is sequentially jointly continuous. The algebra 1^1 of summable scalar sequences with weak topology $\sigma = \sigma(1^1, c_o)$ is a topological algebra in which multiplication (pointwise) is sequentially jointly continuous. Let $e_n = (\delta_{nm})_{m=1}^\infty$. Then (f_n) defined by $f_1 = e_1, f_n = (-1)^{n+1}e_1 + e_n (n \geq 2)$ is a basis which is not Schauder [2]. In fact, $f_1^* = e_1^* + e_2^* - e_3^* + e_4^* - e_5^* + \ldots$, $f_n^* = e_n^*$ $(n \geq 2)$, $f_1^* \not\in c_o$.

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