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ON THE CAYLEY-HAMILTON PROPERTY IN ABELIAN GROUPS

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ABSTRACT. In this paper, the work of Casacuberta and Hilton on the class of abelian fg-like groups is extended. These groups share much in common with the class of finitely generated abelian groups.

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1. INTRODUCTION

A nilpotent group G is said to be finitely generated at every prime (fgp) if for each prime p there is a finitely generated nilpotent group M such that $G_p \simeq M_p$. If there exists a single finitely generated group M which works for all primes, then we say G is fg-like, or more specifically, M-like. Here G_p denotes the p-localization of G. In this paper, we continue the study initiated by Casacuberta and Hilton on properties of fgp groups. In particular, we focus our attention on annihilator properties, in the sense of the Cayley-Hamilton theorem, which an abelian group may satisfy.

Casacuberta and Hilton [2] showed that if A is a special fgp group, the following are equivalent:

(a) A is fg-like;

(b) $\forall\phi : A \to A$, $\exists$ non-zero $F \in \mathbb{Z}[t]$ with $F(\phi)(A) = 0$;

(c) $\forall\phi : A \to A$, $\exists$ monic $F \in \mathbb{Z}[t]$ with $F(\phi)(A) = 0$;

(d) $\forall\phi : A \simeq A$, $\exists$ non-zero $F \in \mathbb{Z}[t]$ with $F(\phi)(A) = 0$;

(e) $\forall\phi : A \simeq A$, $\exists$ demonic $F \in \mathbb{Z}[t]$ with $F(\phi)(A) = 0$.

Here, a nilpotent group G is called special if $TG \rightarrowtail G \twoheadrightarrow FG$ splits on the right, where TG is the torsion subgroup and FG is the torsion-free quotient. For an abelian group A, this is equivalent to $A \simeq TA \oplus FA$. We should also mention that a demonic polynomial is a polynomial whose leading coefficient and constant coefficient are ± 1.

The theorem is proven by showing the following implications:

$$\begin{array}{ccccccc} (a) & \Rightarrow & (c) & \Rightarrow & (e) & & \\ & & \Downarrow & & \Downarrow & & \\ & & (b) & \Rightarrow & (d) & \Rightarrow & (a) \end{array}$$

Only the implication $(d) \Rightarrow (a)$ requires that A be special. Naturally, one should ask whether there exist fgp groups which are non-special (and hence are not fg-like) which satisfy property (d). Casacuberta and Hilton have shown the existence of such a group. In this paper, we show that there does exist a large class of fgp groups which are non-special and which do not satisfy property (d).

We will say that an abelian group A is *Cayley-Hamilton* if it satisfies property (c). We will call A *almost Cayley-Hamilton* if it satisfies property (d). Obviously, Cayley-Hamilton implies almost Cayley-Hamilton; however, it is easy to see that the converse is not true. Thus, let $A = \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}/2$. Then $F(t) = 2t$ annihilates ϕ for any $\phi : A \to A$. However, if we let the i^{th} copy of $\mathbb{Z}/2$ be generated by x_i, then $\phi : A \simeq A$, given by $\phi(x_i) = x_{i+1}$, cannot be annihilated by any polynomial which is not a multiple of 2.

In Section 2, our goal is to show that if A is an abelian group whose torsion-free quotient is $\mathbb{Z}$-like and if, for an infinite number of primes p, TA_p contains either $\mathbb{Z}/p^{n(p)} \oplus \mathbb{Z}/p^{n(p)}$ or $\mathbb{Z}/p^{n(p)} \oplus \mathbb{Z}/p^{n(p)+1}$ as a direct summand, then A is not almost Cayley-Hamilton. In Section 3, we generalize the results of Section 2 by replacing the condition that the torsion-free quotient is $\mathbb{Z}$-like by the condition that the torsion-free quotient is $\mathbb{Z}^k$-like. Of course it is natural to conjecture that the theorem of Casacuberta and Hilton remains true if one considers any fgp groups which are special or have, for an infinite number of primes, non-cyclic torsion components. One should note that our results do not require that A is fgp. We only assume that the torsion-free quotient is $\mathbb{Z}^k$-like. We call such groups *almost* fgp and we describe some properties of these groups in Section 3.

In a forthcoming paper, the author will discuss a generalization of the Cayley-Hamilton property in the class of solvable groups.

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2. ON ALMOST fgp ABELIAN GROUPS OF RANK ONE.

Let A be an abelian group. We denote by [A] the class of the extension $TA \rightarrowtail A \twoheadrightarrow FA$ in the abelian group $\mathrm{Ext}(FA, TA)$. We will sometimes write T for TA and F for FA.

Initially, we will place no restrictions on $T = TA$. However, we will assume throughout that $F = FA$ is $\mathbb{Z}^k$-like for some positive integer k. We will refer to such abelian groups as being *almost* fgp. It is clear that an almost fgp group is fgp if and only if $T_p = TA_p$ is finite for each prime p.

Our first objective is to compute $\mathrm{Ext}(F, T)$ when $k = 1$. To do this, we identify F with a group of pseudo-integers (see [5]). Explicitly, let $F = \langle \frac{1}{p^{k(p)}} |$ all primes p, where $k(p)$ is a non-negative integer for each prime $p\rangle \subseteq \mathbb{Q}$.

LEMMA 1. If F is as above and T is torsion, then $\mathrm{Ext}(F,T) = \dfrac{\prod_p T_p/p^{k(p)}T_p}{\oplus_p T_p/p^{k(p)}T_p}$.

PROOF. Consider the exact sequence $\mathbb{Z} \rightarrowtail F \twoheadrightarrow F/\mathbb{Z} = \oplus_p \mathbb{Z}/p^{k(p)}$.
This sequence induces the following right-exact sequence:

$$\mathrm{Hom}(\mathbb{Z}, T) \xrightarrow{\Delta} \mathrm{Ext}(\oplus_p \mathbb{Z}/p^{k(p)}, T) \twoheadrightarrow \mathrm{Ext}(F, T).$$

However, $\text{Ext}(\bigoplus_p \mathbf{Z}/p^{k(p)}, T) = \prod_p \text{Ext}(\mathbf{Z}/p^{k(p)}, T) = \prod_p T/p^{k(p)}T = \prod_p T_p/p^{k(p)}T_p$ Further, $\text{Hom}(\mathbf{Z}, T) = T = \bigoplus_p T_p$ and $\Delta : \bigoplus_p T_p \to \prod_p T_p/p^{k(p)}T_p$ is the canonical map. Hence $\Delta(T) = \bigoplus_p T_p/p^{k(p)}T_p$. Since the sequence is right-exact, it follows that $\text{Ext}(F,T) = \dfrac{\prod_p T_p/p^{k(p)}T_p}{\bigoplus_p T_p/p^{k(p)}T_p}$.

Q.E.D.

Obviously, if T is almost torsion-free, then $\text{Ext}(F,T) = 0$. Hence, $A = T \oplus F$ and A is special. Of course this result follows from a theorem of Casacuberta and Hilton [1] which states that if A is B-like and T is almost torsion-free, then $\text{Ext}(A,T) \simeq \text{Ext}(B,T)$. We formally state this result as:

THEOREM 2. An almost fgp abelian group of rank one which is almost torsion-free is special.

An application of Lemma 1.5 in Casacuberta and Hilton [2], gives the following corollary:

COROLLARY 3. An almost fgp abelian group of rank one which is almost torsion-free is Cayley-Hamilton (almost Cayley-Hamilton) if and only if T_p is Cayley-Hamilton (almost Cayley-Hamilton) for each prime p.

We now explore property (d) in the class of almost fgp groups of rank one which are not almost torsion-free. We will write $[(\overline{w}_p)]$ for $[A]$ where $w_p \in T_p$, and $\overline{w}_p$ is the coset of w_p modulo $p^{k(p)}T_p$. The next result serves as a prelude for the main theorem of this section.

THEOREM 4. Let A be an abelian group and let F be $\mathbf{Z}$-like. Suppose that, for infinitely many primes p, $T_p \neq 0$ and $\overline{w}_p = 0$. Then A is not almost Cayley-Hamilton.

PROOF. We adapt an argument of Casacuberta and Hilton in [2]. Let $p_1, p_2, \dots$ be an enumeration of the primes with the stated property. Let $\lambda_1, \lambda_2, \dots$ be a sequence of positive integers such that each positive integer occurs infinitely often. As in [2], set $\mu_i = \begin{cases} 1 & \text{if } p_i \mid \lambda_i \\ \lambda_i & \text{if } p_i \nmid \lambda_i \end{cases}$ and define $\phi'_{p_i} : T_{p_i} \to T_{p_i}$ by $\phi'_{p_i}(x) = \mu_i x$. If q is a prime not included in the above enumeration, let $\phi'_q =$ id: $T_q \to T_q$. Let $\phi' = \sum_p \phi'_p : T \to T$. Note that ϕ' is an automorphism of T. Since $\overline{w}_p = 0$ for $p = p_1, p_2, \dots$ and $\phi'_q =$ id for $q \neq p_1, p_2, \dots$ it follows that $\phi'_*[A] = [A]$. Thus ϕ' extends to an endomorphism ϕ of A which induces the identity on F. Thus ϕ is an automorphism of A. Now let G be a non-zero polynomial over $\mathbf{Z}$ with $G(\phi)(A) = 0$. Then $G(\phi')(T) = 0$. Let n be a positive integer. Note that $n = \mu_i$ for infinitely many i, say $j_1, j_2, \dots$. For any given j in the previous sequence, let $x_j \in T_{p_j} - 0$. Then $0 = G(\phi')(x_j) = G(\mu_j)(x_j)$. Since $x_j \neq 0$, it follows that $p_j \mid G(\mu_j)$. Hence for $j = j_1, j_2, \dots$ it follows that $p_j \mid G(n)$. Hence $G(n) = 0$. It follows that $G \equiv 0$ and A is not almost Cayley-Hamilton.

Q.E.D.

COROLLARY 5. Let A be an abelian group and let F be $\mathbf{Z}$-like. Suppose that, for infinitely many primes $p, T_p \neq 0$ while $k(p) = 0$. Then A is not almost Cayley-Hamilton.

PROOF. For the infinite number of primes specified, $\overline{w}_p = 0$. Now apply Theorem 4.

Q.E.D.

We now place conditions on T in order to prove our main result and its corollary.

THEOREM 6. Suppose that for infinitely many p, T_p is not cyclic, and $k(p) \neq 0$. Call this set of primes S. For each $p \in S$, write $T_p = \mathbf{Z}/p^{m(p)} \oplus \mathbf{Z}/p^{n(p)} \oplus$ (finite abelian p-group), where, for each p, $m(p)$ and $n(p)$ are positive integers with $n(p) \geq m(p)$. Let $\langle x_p \rangle = \mathbf{Z}/p^{m(p)}$ and $\langle y_p \rangle = \mathbf{Z}/p^{n(p)}$ and write $w_p = (r_p x_p, q_p y_p, \ldots)$ for some integers r_p and q_p which depend on p. If any one of the following conditions hold, then A is not almost Cayley-Hamilton.

(a) q_p or r_p may be chosen to equal 0 for infinitely many $p \in S$.

(b) q_p, r_p may be chosen so that $|q_p|_p \leq |r_p|_p$ for infinitely many $p \in S$.

(c) q_p, r_p may be chosen so that $|r_p|_p + n(p) \leq |q_p|_p + m(p)$ for infinitely many $p \in S$.

Here $|t|_p$ is the exact power of p which divides t.

PROOF. If either $q_p = 0$ or $r_p = 0$ for infinitely many $p \in S$, then a generalization of the argument in Theorem 4 shows that A is not almost Cayley-Hamilton.

Suppose that (b) holds. Let $S' \subseteq S$ be the set of primes mentioned in (b). Let $p_1, p_2, \ldots$, be an enumeration of the primes in S'. Let $\lambda_1, \lambda_2, \ldots$ be a sequence of positive integers such that each positive integer appears infinitely often. Let $\mu_{p_i} = \begin{cases} 1 & \text{if } p_i | \lambda_i \\ \lambda_i & \text{if } p_i \nmid \lambda_i \end{cases}$. For each prime p in S', define $\psi_p : \mathbf{Z}/p^{n(p)} \to \mathbf{Z}/p^{m(p)}$ so that $q_p \psi_p(y_p) = r_p(1 - \mu_p)x_p$. Condition (b) and the fact that $n(p) \geq m(p)$ guarantee that this is possible. Let $M_p = \begin{bmatrix} \mu_p & \psi_p \\ 0 & 1 \end{bmatrix}$ and let $\phi'_p : T_p \to T_p$ be defined by the matrix $\begin{bmatrix} M_p & 0 \\ 0 & I \end{bmatrix}$. Again, it is easy to see that ϕ'_p is an automorphism of T_p. For $p \notin S'$, let $\phi'_p =$id. Let $\phi' = \sum_p \phi'_p : T \to T$. We observe that ϕ' is an automorphism of T. The formula $q_p \psi_p(y_p) = r_p(1 - \mu_p)x_p$ guarantees $\phi'_p w_p = w_p$. Hence $\phi'_*[A] = [A]$. Thus ϕ' extends to an automorphism $\phi : A \to A$ which induces the identity on F.

Finally, note that $\phi'_p(x_p) = \mu_p x_p$. As in the proof of Theorem 4, there does not exist a non-zero polynomial over $\mathbf{Z}$ which annihilates ϕ'. Hence there does not exist such a polynomial which annihilates ϕ. Hence A is not almost Cayley-Hamilton. The proof using condition (c) is similar and depends on defining a homomorphism $\psi'_p : \mathbf{Z}/p^{m(p)} \to \mathbf{Z}/p^{n(p)}$ with properties similar to those of ϕ'_p above.

COROLLARY 7. Suppose that, for infinitely many p, T_p contains either $\mathbf{Z}/p^{n(p)} \oplus \mathbf{Z}/p^{n(p)}$ or $\mathbf{Z}/p^{n(p)} \oplus \mathbf{Z}/p^{n(p)+1}$ as a direct summand. Then A is not almost Cayley-Hamilton.

Now let $T_p = \mathbf{Z}/p = \langle w_p \rangle$ for each prime p. Let $F = \langle \frac{1}{p} |$ all primes $p \rangle$ and let A be the abelian group defined by $[A] = [(w_p)] \in$ Ext (F, T). It was shown by Casacuberta and Hilton [2] that A is Cayley-Hamilton. The following lemma shows that $A \otimes A$ satisfies the hypothesis of Corollary 7. Thus, the tensor product of Cayley-Hamilton groups need not be almost Cayley-Hamilton.

LEMMA 8. Let A and B be almost fgp groups where FA is $\mathbf{Z}^k$-like and FB is $\mathbf{Z}^l$-like. Then $A \otimes B$ is almost fgp where $F(A \otimes B)$ is $\mathbf{Z}^{kl}$-like. Moreover, $T(A \otimes B)_p = TA_p^l \oplus TB_p^k \oplus (TA_p \otimes TB_p)$.

PROOF. Note that $A_p = \mathbf{Z}_p^k \oplus TA_p$ and $B_p = \mathbf{Z}_p^l \oplus TB_p$. Then $(A \otimes B)_p = A_p \otimes B_p =$

$$(\mathbf{Z}_p^k \oplus TA_p) \otimes (\mathbf{Z}_p^l \oplus TB_p) = (\mathbf{Z}_p^k \otimes \mathbf{Z}_p^l) \oplus (TA_p \otimes \mathbf{Z}_p^l) \oplus (\mathbf{Z}_p^k \otimes TB_p) \oplus (TA_p \otimes TB_p).$$
$$= \mathbf{Z}_p^{kl} \oplus TA_p^l \oplus TB_p^k \oplus (TA_p \otimes TB_p).$$

The result follows since $T(A \otimes B)_p = T((A \otimes B)_p)$ and $F(A \otimes B)_p = F((A \otimes B)_p)$.

Q.E.D.

An amusing consequence of Lemma 8 is that if A and B are almost fgp and neither A or B is torsion, then, if $A \otimes B$ is fg-like, it follows that A and B are fg-like.

3. ON ALMOST fgp ABELIAN GROUPS.

We now seek to generalize the results of Section 2. Given a $\mathbf{Z}^k$-like group F, we choose a reduced representation R_* for F in the sense of Casacuberta and Hilton [1]. Then we may replace F by the subgroup of $\mathbf{Q}^k$ generated by the column vectors of R_p for each prime p. Casacuberta and Hilton have shown [1] that F contains $\mathbf{Z}^k$. The quotient $\overline{F} = F/\mathbf{Z}^k$ is a torsion group. Moreover, $\overline{F}_p$ is generated sby the cosets containing the column vectors of R_p.

EXAMPLE 9.. Let $R_p = \begin{bmatrix} 1/p & 1 \\ 0 & 1/p \end{bmatrix}$. Then $\overline{F}_p = \langle (1, \frac{1}{p}) + \mathbf{Z}^2, (\frac{1}{p}, 0) + \mathbf{Z}^2 \rangle \simeq \mathbf{Z}/p \oplus \mathbf{Z}/p$.

EXAMPLE 10.. Let $R_p = \begin{bmatrix} 1/p & 1/p^2 \\ 0 & 1 \end{bmatrix}$. Then $\overline{F}_p = \langle (\frac{1}{p^2}, 1) + \mathbf{Z}^2 \rangle \simeq \mathbf{Z}/p^2$.

In general, we may write $\overline{F}_p \simeq \mathbf{Z}/p^{n_1(p)} \oplus \mathbf{Z}/p^{n_2(p)} \oplus \cdots \oplus \mathbf{Z}/p^{n_k(p)}$ where $n_1(p) \leq n_2(p) \leq \cdots \leq n_k(p)$ and some of the factors may be trivial.

LEMMA 11. If T is torsion and F is $\mathbf{Z}^k$-like, then

$$\operatorname{Ext}(F, T) = \frac{\prod_p [T_p/p^{n_1(p)}T_p \oplus \cdots \oplus T_p/p^{n_k(p)}T_p]}{\bigoplus_p [T_p/p^{n_1(p)}T_p \oplus \cdots \oplus T_p/p^{n_k(p)}T_p]}.$$

PROOF.. Consider the exact sequence $\mathbf{Z}^k \rightarrowtail F \twoheadrightarrow \overline{F} = \bigoplus_p \overline{F}_p$.

This sequence induces the following right-exact sequence:

$$\operatorname{Hom}(\mathbf{Z}^k, T) \xrightarrow{\Delta} \operatorname{Ext}(\bigoplus_p \overline{F}_p, T) \twoheadrightarrow \operatorname{Ext}(F, T).$$

However,

$$\begin{aligned} \operatorname{Ext}(\bigoplus_p \overline{F}_p, T) = \prod_p \operatorname{Ext}(\overline{F}_p, T) &= \prod_p \operatorname{Ext}(\mathbf{Z}/p^{n_1(p)} \oplus \cdots \oplus \mathbf{Z}/p^{n_k(p)}, T) \\ &= \prod_p [\operatorname{Ext}(\mathbf{Z}/p^{n_1(p)}, T) \oplus \cdots \oplus \operatorname{Ext}(\mathbf{Z}/p^{n_k(p)}, T)]. \\ &= \prod_p [T/p^{n_1(p)}T \oplus \cdots \oplus T/p^{n_k(p)}T] \\ &= \prod_p [T_p/p^{n_1(p)T_p} \oplus \cdots \oplus T_p/p^{n_k(p)}T_p]. \end{aligned}$$

Also $\operatorname{Hom}(\mathbf{Z}^k, T) = T^k = (\bigoplus_p T_p)^k$ and $\Delta : (\bigoplus_p T_p)^k \to \prod_p [T_p/p^{n_1(p)}T_p \oplus \cdots \oplus \oplus T_p/p^{n_k(p)}T_p]$ is the canonical map.

Thus $\Delta((\bigoplus_p T_p)^k) = \bigoplus_p [T_p/p^{n_1(p)T_p} \oplus \cdots \oplus T_p/p^{n_k(p)}T_p]$. Since the above sequence is right exact, the result follows.

Q.E.D.

Once again, it is clear from Lemma 11 that if T is almost torsion-free, then A is special. Hence we have the following generalizations of Theorem 2 and Corollary 3.

THEOREM 12.. An almost fgp, almost torsion-free group is special.

COROLLARY 13.. An almost fgp, almost torsion-free group is Cayley-Hamilton (almost Cayley-Hamilton) if and only if T_p is Cayley-Hamilton (almost Cayley-Hamilton) for each prime p.

We also obtain generalizations of Theorem 4 and Corollary 5.

THEOREM 14.. Suppose that, for infinitely many primes $p, w_p = 0$ while $T_p \neq 0$. Then A is not almost Cayley-Hamilton.

COROLLARY 15.. Suppose that, for infinitely many primes p, $n_1(p) \geq \log_p \exp(T_p)$ and $T_p \neq 0$. Then A is not almost Cayley-Hamilton.

Now we seek to extend Corollary 7. Before doing this we need to place an additional hypothesis on F.

DEFINITION 16.. Let R_* be a reduced representation for F. R_* is called *collapsible* if for each prime p, $\overline{F}_p$ is cyclic. In this case, $\overline{F}_p \simeq \mathbf{Z}/p^{n_k(p)}$. If F is an arbitrary $\mathbf{Z}^k$-like group, then we call F *collapsible* if there exists a reduced representation of F which is collapsible.

Note that all $\mathbf{Z}$-like groups are collapsible. As a result, the following theorem generalizes Corollary 7.

THEOREM 17. Suppose that F is $\mathbf{Z}^k$-like for $k \geq 1$ and that F is collapsible. Suppose that, for an infinite number of primes p, T_p contains either $\mathbf{Z}/p^{n(p)} \oplus \mathbf{Z}/p^{n(p)}$ or $\mathbf{Z}/p^{n(p)} \oplus \mathbf{Z}/p^{n(p)+1}$ as a direct summand. Then A is not almost Cayley-Hamilton.

We close by mentioning that the class of almost fgp abelian groups is a somewhat well behaved class of abelian groups. It is straight forward to show that the class is closed under the formation of quotients, subgroups, tensor products, torsion products, and homology. However, in contrast to the class of fgp abelian groups, this class is not a Serre class because it fails to satisfy the condition of closure under extensions. For example, in the exact sequence $\mathbf{Z} \rightarrowtail \mathbf{Q} \twoheadrightarrow \mathbf{Q}/\mathbf{Z}$ it is clear that $\mathbf{Z}$ and $\mathbf{Q}/\mathbf{Z}$ are almost fgp; however, $\mathbf{Q}$ is clearly not almost fgp. We are presently exploring properties of nilpotent groups whose torsion-free quotient is fg-like.

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