Internat. J. Math. & Math. Sci.
VOL. 15 NO. 1 (1992) 47-56

ON THE DEGREE OF APPROXIMATION OF THE HERMITE AND HERMITE-FEJER INTERPOLATION

J. PRASAD

Department of Mathematics
California State University
Los Angeles, California 90032 U.S.A.

(Received January 25, 1991 and in revised form May 22, 1991)

ABSTRACT. Here we find the order of convergence of the Hermite and Hermite-Fejér interpolation polynomials constructed on the zeros of $(1-x^2)P_n(x)$ where $P_n(x)$ is the Legendre polynomial of degree n with normalization $P_n(1)=1$.

KEY WORDS AND PHRASES. Zeros, modulus of continuity, interior points, interpolation process, best approximation.

1980 AMS SUBJECT CLASSIFICATION CODE. 41A25.

1. INTRODUCTION.

Let

$$-1 = x_{n+1} < x_n < \ldots < x_1 < x_0 = 1 \tag{1.1}$$

be the $n+2$ distinct zeros of $(1-x^2)P_n(x)$ where $P_n(x)$ is the Legendre polynomial of degree n with normalization $P_n(1)=1$. Let f be a given function on $[-1,1]$. Let $Q_n(f,x)$ be the unique polynomial of degree $\le 2n+1$ such that

$$Q_n(f,x_k) = f(x_k),\ \ Q_n(f,\pm 1) = f(\pm 1),\ \ Q_n'(f,x_k) = 0,\ \ k = 1,2,\ldots,n. \tag{1.2}$$

Then it is known [11] that

$$Q_n(f,x) = f(-1)\,\frac{1-x}{2}P_n^2(x) + f(1)\,\frac{1+x}{2}P_n^2(x) + \sum_{k=1}^{n} f(x_k)\,\frac{1-x^2}{1-x_k^2}\,1_k^2(x) \tag{1.3}$$

where

$$1_k(x) = \frac{P_n(x)}{(x-x_k)P_n'(x_k)} \tag{1.4}$$

is the fundamental polynomial of the Lagrange interpolation. According to a well-known result of Szasz [11]

$$\lim_{n\to\infty} Q_n(f,x) = f(x),$$

uniformly on $[-1,1]$, for every f continuous there. A quantitative version of Szasz's result was given by Prasad and Saxena [8] who showed that

$$|Q_n(f,x)-f(x)| \leq c_1 n^{-1}\sum_{k=1}^{n} w_f\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right) \tag{1.5}$$

where w_f is the modulus of continuity of f on $[-1,1]$ on c_1 (later on $c_2, c_3 \ldots$) is a positive absolute constant. Prasad and Varma [9] further improved (1.5) by proving that

$$|Q_n(f,x)-f(x)| \leq c_2 n^{-1}\sum_{k=1}^{n} w_f\left(\frac{\sqrt{1-x^2}}{k}\right). \tag{1.6}$$

Our aim is here to consider the interpolation process which requires that the derivative of the polynomial vanishes not only at the interior points $x_k, k=1,2,\ldots,n$, but also at the end points -1 and 1. As we will see, the situation is quite different in this case. Let us denote by $R_n(f,x)$ the unique polynomial of degree $\leq 2n+3$ satisfying the conditions

$$R_n(f,x_k)=f(x_k),\ \ R_n(f,\pm 1)=f(\pm 1),\ \ R_n'(f,x_k)=0,\ \ R_n'(f,\pm 1)=0,\ \ k=1,2,\ldots,n. \tag{1.7}$$

Then from (1.2), (1.3) and (1.7) it follows that

$$\begin{aligned} R_n(f,x) &= Q_n(f,x)+(1-x)\left[\frac{(1+x)P_n(x)}{2}\right]^2 Q_n'(f,1) \\ &\quad -(1+x)\left[\frac{(1-x)P_n(x)}{2}\right]^2 Q_n'(f,-1). \end{aligned} \tag{1.8}$$

Bojanic, Varma and Vertesi [3] proved the following:

THEOREM A. Let $f \in C[-1,1]$. In the case when $\alpha \in [-\frac{1}{2},\frac{1}{2})$ the necessary and sufficient conditions for

$$\lim_{n\to\infty} \| R_n^{(\alpha,\alpha)}(f,x)-f(x) \| = 0 \tag{1.9}$$

is given by

$$\lim_{n\to\infty}\int_{-1}^{1}\left(R_n^{(\alpha,\alpha)}(f,x)-f(x)\right)dx=0$$

and

$$\lim_{n\to\infty}\int_{-1}^{1}x\left(R_n^{(\alpha,\alpha)}(f,x)-f(x)\right)dx=0.$$

If $\alpha \in [\frac{1}{2},2)$, (1.9) holds true for arbitrary $f \in C[-1,1]$ (i.e., no conditions are needed).

In the case $\alpha \in [p-1,p)$, $p \geq 3$ (p integer) the necessary and sufficient conditions for the validity of (1.9) is given by

$$\left[R_n^{(\alpha,\alpha)}(f,x)\right]^{(r)}_{x=\pm 1}=o(n^{2r}), r=1,2,\ldots,p-1.$$

Here $R_n^{(\alpha,\alpha)}(f,x)$ is the polynomial of degree $\leq 2n+3$ satisfying the interpolatory conditions (1.7) on the zeros of ultraspherical polynomial.

For $R_n(f,x)$ satisfying the interpolatory conditions (1.7) on the zeros of $(1-x^2)P_n(x)$ we prove the following:

THEOREM 1. Let $f \in C[-1,1]$ and let $R_n(f,x)$ be the Hermite-Fejer interpolation polynomial of degree $\leq 2n+3$ defined by (1.7). Then for all $x \in [-1,1]$

$$|R_n(f,x) - f(x)| \leq \frac{c_3}{n}\sum_{k=1}^{n} w_f\left(\frac{\sqrt{1-x^2}}{k}\right) + c_4\sqrt{1-x^2}\left[\frac{1}{n} + \sum_{k=1}^{n} w_f\left(\frac{1}{k^2}\right)\right]. \tag{1.10}$$

From (1.10) it is evident that the sequence $\{R_n(f,x)\}$ converges to $f(x)$ at the end points -1 and 1. Also, if $f \in \mathrm{Lip}\sigma$, $1/2 < \sigma < 1$, then from (1.10) it follows that

$$|R_n(f,x) - f(x)| \leq \frac{c_5}{n}\sum_{k=1}^{n} \frac{(1-x^2)^{\sigma/2}}{k^{\sigma}} + c_6\sqrt{1-x^2}\, n^{1-2\sigma}$$

$$\leq c_7(1-x^2)^{\sigma/2} n^{-\sigma} + c_6\sqrt{1-x^2}\, n^{1-2\sigma}$$

$$\leq c_8(1-x^2)^{\frac{\sigma}{2}} n^{1-2\sigma}$$

Thus, $\{R_n(f,x)\}$ converges to $f(x)$ for $-1 \leq x \leq 1$ if $f \in \mathrm{Lip}\sigma$, $\frac{1}{2} < \sigma < 1$. Next, we show that there is a $f \in C[-1,1]$ for which $\{R_n(f,x)\}$ diverges at $x = 0$. More precisely we prove the following:

THEOREM 2. The Hermite-Fejér interpolation process $\{R_n(f,x)\}$ for the function $f(x) = -(1-x^2)^{\sigma}$, $0 < \sigma \leq \frac{1}{2}$ with the nodes (1.1) diverges at the point $x = 0$.

This result is similar to a result of Berman [1] who proved that the Hermite-Fejér interpolation process $\{H_n(f,x)\}$ constructed for $f(x) = |x|$ with

$$x_k = \cos\frac{2k-1}{2n}\pi, \quad k = 1,2,\ldots,n, x_0 = 1, \quad x_{n+1} = -1; \quad n = 1,2,\ldots,$$

diverges at $x = 0$.

Let $F_n(f,x)$ be the unique polynomial of degree $\leq 2n+1$ satisfying the conditions

$$\begin{gathered} F_n(f,-1) = f(-1), \quad F_n(f,1) = f(1), \\ F_n(f,x_k) = f(x_k), \quad F_n'(f,x_k) = f'(x_k), \quad k = 1,2,\ldots n, \end{gathered} \tag{1.11}$$

where $x_k's$ are given by (1.1). Then we see that

$$\begin{aligned} F_n(f,x) &= Q_n(f,x) + \sum_{k=1}^{n} f'(x_k)\frac{1-x^2}{1-x_k^2}(x-x_k)1_k^2(x) \\ &= \sum_{k=0}^{n+1} f(x_k)h_k(x) + \sum_{k=1}^{n} f'(x_k)\sigma_k(x) \end{aligned} \tag{1.12}$$

where

$$h_0(x) = \frac{1+x}{2}P_n^2(x), \quad h_{n+1}(x) = \frac{1-x}{2}P_n^2(x) \tag{1.13}$$

and

$$h_k(x) = \frac{1-x^2}{1-x_k^2}1_k^2(x), \quad \sigma_k(x) = (x-x_k)h_k(x), \quad k = 1,2,\ldots,n. \tag{1.14}$$

For the polynomials $F_n(f,x)$ we prove the following:

THEOREM 3. Let f be defined and have a continuous derivative f' on $[-1,1]$. Then for the Hermite interpolation polynomial $F_n(f,x)$ of degree $\leq 2n+1$ defined by (1.11) we have for all $x \in [-1,1]$,

$$|F_n(f,x) - f(x)| \leq c_9 \frac{\log n}{n} E_{2n}(f')$$

where $E_{2n}(f')$ is the best approximation of $f'(x)$ by polynomials of degree at most $2n$.

Now, if we compare the zeros of $(1-x^2)P_n(x)$ with the zeros of $(1-x^2)T_n(x)$, the n^{th} degree Tchebycheff polynomial of the first kind, we find that they are equally good as far as the convergence and the order of convergence of the Hermite and the Hermite Fejér interpolation is concerned.

2. PRELIMINARIES. In this section we state a few known results which we shall use later on.

From [5] we have for $-1 \leq x \leq 1$,

$$P_n^2(x) + \sum_{k=1}^{n} h_k(x) = 1. \tag{2.1}$$

Further, due to Fejér [7] we know that

$$\sum_{k=1}^{n} \frac{1}{(1-x_k^2)[P_n'(x_k)]^2} = 1. \tag{2.2}$$

Also, from Szegö [12] we have

$$(1-x^2)^{1/4} |P_n(x)| \leq \left(\tfrac{2}{\pi}\right)^{1/2} n^{-1/2}, \quad -1 \leq x \leq 1, \tag{2.3}$$

$$(1-x^2) > \left(k-\tfrac{1}{2}\right)^2 \left(n+\tfrac{1}{2}\right)^{-2}, \quad k = 1,2,\ldots,[\tfrac{n}{2}], \tag{2.4}$$

$$(1-x_k^2) > \left(n-k+\tfrac{1}{2}\right)^2 \left(n+\tfrac{1}{2}\right)^{-2}, \quad k = [\tfrac{n}{2}]+1,\ldots,n, \tag{2.5}$$

$$|P_n'(x_k)| \sim k^{-3/2} n^2, \quad k = 1,2,\ldots[\tfrac{n}{2}], \tag{2.6}$$

$$|P_n'(x_k)| \sim (n+1-k)^{-3/2} n^2, \quad k = [\tfrac{n}{2}]+1,\ldots,n, \tag{2.7}$$

$$\begin{aligned} |P_n(0)| &= \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (n-1)}{2\cdot 4\cdot \cdots \cdot (n-2)n} \\ &> \tfrac{1}{3} n^{-1/2} \end{aligned} \tag{2.8}$$

and

$$\frac{(k-1/2)\pi}{n+1/2} < \Theta_k < \frac{k\pi}{n+1/2}, \quad k = 1,2,\ldots,n, \quad x = \cos\Theta \text{ and } x_k = \cos\Theta_k. \tag{2.9}$$

Further, from [9] we also have for $-1 \leq x \leq 1$ and $k = 1,2,\ldots,n$,

$$\frac{(1-x^2)^{1/4} |P_n(x)|}{(1-x_k^2)^{3/4} |P_n'(x_k)|} \leq \frac{c_{10}}{n}, \quad n \geq 2, \tag{2.10}$$

and

$$\frac{(1-x^2)^{1/4}}{(1-x_k^2)^{1/4}} |l_k(x)| \leq c_{11}. \tag{2.11}$$

Next, from [6] it follows that for $-1 \le x \le 1$,

$$|1_k(x)| \le c_{12}, \quad k = 1,2,\dots,n. \tag{2.12}$$

3. SOME LEMMAS. In this section we state and prove a few lemmas which will enable us to prove the theorems.

First, we assume x_j to be that zero of $P_n(x)$ which is nearest to x. From (2.9) and using the fact that x_j is the nearest zero to x we get

$$\frac{1}{\sin\frac{|\Theta-\Theta_k|}{2}} \le \frac{2n+1}{2i-1}, \quad k \ne j, \quad k = j \pm i. \tag{3.1}$$

Also, we note that

$$\sin\Theta \le \sin\Theta + \sin\Theta_k \le 2\sin\left(\frac{\Theta+\Theta_k}{2}\right). \tag{3.2}$$

LEMMA 1. If the polynomials $Q_n(f,x)$ and $R_n(f,x)$ are defined by (1.2) and (1.7), respectively, then

$$R_n(f,x) = Q_n(f,x) + \frac{x(1-x^2)}{4} P_n^2(x)\,[f(1)-f(-1)]$$
$$+\tfrac{1}{2}(1-x^2)(1+x)\,P_n^2(x)\sum_{k=1}^{n}\frac{[f(1)-f(x_k)]}{(1-x_k^2)(1-x_k)^2[P_n'(x_k)]^2}$$
$$+\tfrac{1}{2}(1-x^2)(1-x)\,P_n^2(x)\sum_{k=1}^{n}\frac{[f(-1)-f(x_k)]}{(1-x_k^2)(1+x_k)^2[P_n'(x_k)]^2}$$

PROOF. From (2.1) it follows that

$$P_n'(1) = \sum_{k=1}^{n}\frac{1}{(1-x_k^2)(1-x_k)^2[P_n'(x_k)]^2}. \tag{3.3}$$

Also, it is easy to see that

$$P_n(-1)P_n'(-1) = -P_n'(1). \tag{3.4}$$

Consequently, from (1.3, (3.3) and (3.4) we obtain

$$Q_n'(f,1) = \tfrac{1}{2}[f(1)-f(-1)] + 2\sum_{k=1}^{n}\frac{[f(1)-f(x_k)]}{(1-x_k^2)(1-x_k)^2[P_n'(x_k)]^2}. \tag{3.5}$$

and

$$Q_n'(f,-1) = \tfrac{1}{2}[f(1)-f(-1)] - 2\sum_{k=1}^{n}\frac{[f(-1)-f(x_k)]}{(1-x_k^2)(1+x_k)^2[P_n'(x_k)]^2}. \tag{3.6}$$

Substitution into (1.8) yields the desired result.

LEMMA 2. Let f be a continuous function on $[-1,1]$ and let x_k, $k = 1,2,\dots,n$, be the zeros of $P_n(x)$, the n^{th} degree Legendre polynomial, then

$$\left|\sum_{k=1}^{n}\frac{[f(\pm 1)-f(x_k)]}{(1-x_k^2)(1\mp x_k)^2[P_n'(x_k)]^2}\right| \le c_{13}\, n\sum_{k=1}^{n} w_f\left(\frac{1}{k^2}\right).$$

PROOF. It is clearly sufficient to consider one choice of signs. Let us put $m = [\frac{n}{2}]$ and consider

$$| \Delta_n(f) | = \Big| \sum_{k=1}^{n} \frac{[f(1)-f(x_k)]}{(1-x_k^2)\,(1-x_k)^2\,[P_n'(x_k)]^2} \Big|$$

$$\leq \sum_{k=1}^{m} \frac{| f(1)-f(x_k) |}{(1-x_k^2)\,(1-x_k)^2\,[P'_n(x_k)]^2}$$

$$+ \sum_{k=m+1}^{n} \frac{| f(1)-f(x_k) |}{(1-x_k^2)\,(1-x_k)^2\,[P_n'(x_k)]^2}$$

$$= I_1 + I_2 \,. \tag{3.7}$$

First, we estimate I_2. On using (2.2) we obtain

$$I_2 \leq w_f(2) \sum_{k=m+1}^{n} \frac{1}{(1-x_k^2)\,[P_n'(x_k)]^2} \tag{3.8}$$

$$\leq w_f(2) \sum_{k=1}^{n} \frac{1}{(1-x_k^2)\,[P_n'(x_k)]^2}$$

$$\leq w_f(2)\,.$$

Next, we consider

$$I_1 = \sum_{k=1}^{m} \frac{1}{(1-x_k^2)\,(1-x_k)^2\,[P_n'(x_k)]^2}$$

$$\leq \sum_{k=1}^{m} \frac{w_f(1-x_k)}{1-x_k^2}\left[\frac{1}{(1-x_k)\,P_n'(x_k)}\right]^2 . \tag{3.9}$$

Since

$$1 - x_k = 1 - \cos\Theta_k = 2\sin^2\frac{\Theta_k}{2}$$

hence on using (2.9) we see that

$$\frac{k^2}{2(n+1)^2} \leq 1 - x_k \leq \frac{20k^2}{(n+1)^2}\,. \tag{3.10}$$

Thus from (3.9), (3.10), (2.4) and (2.6) it follows that

$$I_1 \leq c_{14} n^2 \sum_{k=1}^{n} \frac{1}{k^3}\, w_f\left(\frac{k^2}{(n+1)^2}\right). \tag{3.11}$$

Consequently, from (3.7), (3.8) and (3.11) we obtain

$$| \Delta_n(f) | \leq c_{15}\, n^2 \sum_{k=1}^{n} \frac{1}{k^3}\, w_f\left(\frac{k^2}{(n+1)^2}\right). \tag{3.12}$$

Now following the same lines as in [3] we get

$$\sum_{k=1}^{n} \frac{1}{k^3}\, w_f\left(\frac{k^2}{(n+1)^2}\right) \leq \frac{2}{n+1} \sum_{k=1}^{n} w_f\left(\frac{1}{k^2}\right). \tag{3.13}$$

Hence from (3.12) and (3.13) we conclude that

$$| \Delta_n(f) | \le c_{16}\, n \sum_{k=1}^{n} w_f\left(\frac{1}{k^2}\right).$$

LEMMA 3. If $-1 \le x \le 1$ then

$$\sum_{k=1}^{n} \frac{|\sigma_k(x)|}{\sqrt{1-x_k^2}} \le c_{17}\, \frac{\log n}{n}.$$

PROOF. From (1.14) we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{|\sigma_k(x)|}{\sqrt{1-x_k^2}} &= \sum_{k=1}^{n} \frac{(1-x^2)\,|x-x_k|}{(1-x_k^2)^{3/2}}\, 1_k^2(x) \\ &= \frac{(1-x^2)\,|x-x_j|}{(1-x_j^2)^{3/2}}\, 1_j^2(x) + \sum_{k\ne j} \frac{(1-x^2)\,|x-x_k|}{(1-x_k^2)^{3/2}}\, 1_k^2(x) = J_1 + J_2\,.\end{aligned} \tag{3.14}$$

If x_j is the zero of $P_n(x)$ which is nearest to x then one can easily see that

$$\frac{(1-x^2)^{1/2}}{(1-x_j^2)^{1/2}} \le c_{18}\,. \tag{3.15}$$

Now from (2.10), (2.11), (2.12) and (3.15) it follows that

$$\begin{aligned}J_1 &= \frac{(1-x^2)\,|x-x_j|}{(1-x_k^2)^{3/2}}\, 1_j^2(x) \\ &= \left[\frac{(1-x^2)^{1/4}\,|P_n(x)\}}{(1-x_j^2)^{3/4}\,|P_n'(x_j)\,|}\right]\left[\frac{(1-x^2)^{1/4}}{(1-x_j^2)^{1/4}}\,|\,1_j(x)\,|\right]\left[\frac{(1-x^2)^{1/2}}{(1-x_j^2)^{1/2}}\right] \\ &\le c_{19} n^{-1}\,.\end{aligned} \tag{3.16}$$

Next, on using (2.10), (3.2) and (3.1) we obtain

$$\begin{aligned}J_2 &= \sum_{k\ne j} \frac{(1-x^2)\,|x-x_k|}{(1-x_k^2)^{3/2}}\, 1_k^2(x) \\ &\le c_{20} n^{-2} \sum_{k\ne j} \frac{(1-x^2)^{1/2}}{|x-x_k|} \\ &\le c_{20} n^{-2} \sum_{k\ne j} \frac{1}{\sin\frac{|\Theta-\Theta_k|}{2}} \\ &\le c_{21}\, \frac{\log n}{n}\,.\end{aligned} \tag{3.17}$$

Consequently from (3.14), (3.16) and (3.17) it follows that

$$\sum_{k=1}^{n} \frac{|\sigma_k(x)|}{\sqrt{1-x_k^2}} \le c_{22}\, \frac{\log n}{n}$$

which yields the lemma.

4. PROOF OF THEOREM 1. Theorem 1 is now a simple consequence of Lemma 1 and Lemma 2. Due to Lemma 1 we have for $-1 \le x \le 1$,

$$| R_n(f,x) - f(x) | \le | Q_n(f,x) - f(x) | + (1-x^2)P_n^2(x)[| f(1) | + | f(-1) |]$$

$$+ (1-x^2)P_n^2(x) | \sum_{k=1}^{n} \frac{[f(1) - f(x_k)]}{(1-x_k^2)\,(1-x_k)^2\,[P_n'(x_k)]^2} |$$

$$+ (1-x^2)P_n^2(x) | \sum_{k=1}^{n} \frac{[f(-1) - f(x_k)]}{(1-x_k^2)\,(1+x_k)^2\,[P_n'(x_k)]^2} | .$$

Using (1.6), the inequality (2.3), and Lemma 2, we find that

$$| R_n(f,x) - f(x) \le \frac{c_2}{n} \sum_{k=1}^{n} w_f\left(\frac{\sqrt{1-x^2}}{k}\right) + \frac{\sqrt{1-x^2}[| f(1) | + | f(-1) |]}{n}$$

$$+ c_{23}\sqrt{1-x^2} \sum_{k=1}^{n} w_f\left(\frac{1}{k^2}\right)$$

and Theorem 1 follows.

5. PROOF OF THEOREM 2. Let $f(x) = -(1-x^2)^\sigma$, $0 < \sigma \le 1/2$, then we get from (1.3) for $-1 \le x \le 1$,

$$Q_n(f,x) = \sum_{k=1}^{n} f(x_k) \frac{1-x^2}{1-x_k^2} \left[\frac{P_n(x)}{(x-x_k)\,P_n'(x_k)}\right]^2 .$$

Since f is even and x_k are symmetrically situated around 0, $Q_n(f,x)$ is even. Thus $Q_n'(f,x)$ is odd so that $Q_n'(f,-1) = -Q_n'(f,1)$ and we have

$$R_n(f,x) = Q_n\,(f,x) + 1/2\,(1-x^2)\,P_n^2(x)\,Q_n'(f,1) . \tag{5.1}$$

But

$$Q_n'(f,1) = 2\sum_{k=1}^{n} (1-x_k^2)^{\sigma-1} \left[\frac{1}{(1-x_k)\,P_n'(x_k)}\right]^2$$

$$\ge 2\sum_{k=1}^{m} (1-x_k^2)^{\sigma-1} \left[\frac{1}{(1-x_k)\,P_n'(x_k)}\right]^2 , \text{ where } m = [\tfrac{n}{2}],$$

$$\ge 2\sum_{k=1}^{m} (1-x_k^2)^{\sigma-3}[P_n'(x_k)]^{-2} . \tag{5.2}$$

For $k = 1, 2, ..., m, 0 < \Theta_k = \cos^{-1}x_k < \frac{\pi}{2}$. Thus, $1 - x_k^2 = \sin^2\Theta_k \sim \Theta_k^2 \sim \frac{k^2}{n^2}$. Hence on using (2.6) we have from (5.2),

$$Q_n'(f,1) \ge c_{24} n^{2-2\sigma} \sum_{k=1}^{m} \frac{1}{k^{3-2\alpha}} \ge c_{24} n^{2\sigma - 2} . \tag{5.3}$$

Consequently from (5.1), (5.3) and (2.8) we obtain

$$\limsup_{n\to\infty} | R_n(f,0) - Q_n(f,0) | \ge c_{24} \limsup_{n\to\infty} n^{1-2\sigma} .$$

Since $Q_n(f,0) \to f(0) = -1$, $R_n(f,0) \not\to f(0)$ if $0 < \sigma \le 1/2$.

6. PROOF OF THEOREM 3. One can easily see that

$$F_n(f,x)-f(x)=F_n(f,x)-F_n(G_{2n+1},x)+G_{2n+1}(x)-f(x), \tag{6.1}$$

where $G_{2n+1}(x)$ is the polynomial of the best approximation of $f(x)$ and $F_n(f,x)$ is given by (1.12). Thus from (6.1) we obtain

$$|F_n(f,x)-f(x)| \le |F_n(f,x)-F_n(G_{2n+1},x)| + |G_{2n+1}(x)-f(x)| = u_1+u_2\,. \tag{6.2}$$

Now, from the definition of $G_{2n+1}(x)$ it follows that for $-1 \le x \le 1$,

$$|G_{2n+1}(f,x)-f(x)| \le E_{2n+1}(f), \tag{6.3}$$

where $E_{2n+1}(f)$ is the best approximation of $f(x)$. So owing to (6.3) we have for $-1 \le x \le 1$,

$$u_2 \le E_{2n+1}(f)\,. \tag{6.4}$$

Next, we consider

$$\begin{aligned} u_1 &= |F_n(f,x)-F_n(G_{2n+1},x)| \\ &\le \sum_{k=0}^{n+1} |f(x_k)-G_{2n+1}(x_k)|\, h_k(x) \\ &+ \sum_{k=1}^{n} |f'(x_k)-G'_{2n+1}(x_k)|\, |\sigma_k(x)| \\ &= u_1^* + u_2^*\,. \end{aligned} \tag{6.5}$$

Again due to (6.3) and (2.1) we have for $-1 \le x \le 1$,

$$\begin{aligned} u_1^* &\le E_{2n+1}(f)\sum_{k=0}^{n+1} h_k(x) \\ &\le E_{2n+1}(f)\,. \end{aligned} \tag{6.6}$$

Further, on using a theorem of J. Czipszer and G. Freud [4] and Corollary 1.44 of T.J. Rivlin [10], p. 23, it follows that

$$\sqrt{1-x_k^2}\,|f'(x_k)-G'_{2n+1}(x_k)| \le 40E_{2n}(f')\,. \tag{6.7}$$

Hence (6.7) and Lemma 3 yield

$$\begin{aligned} u_2^* &\le 40E_{2n}(f')\sum_{k=1}^{n} \frac{|\sigma_k(x)|}{\sqrt{1-x_k^2}} \\ &\le c_{25}\,\frac{\log n}{n}\,E_{2n}(f')\,. \end{aligned} \tag{6.8}$$

Consequently from (6.2), (6.4), (6.5), (6.6) and (6.8) we obtain for $-1 \le x \le 1$,

$$|F_n(f,x)-f(x)| \le 2E_{2n+1}(f)+c_{25}\,\frac{\log n}{n}\,E_{2n}(f')\,. \tag{6.9}$$

But due to Rivlin [10], p.23, we have

$$E_{2n+1}(f) \le \frac{6}{2n+1}\,E_{2n}(f')\,. \tag{6.10}$$

Hence , from (6.9) and (6.10) the theorem follows.

REFERENCES

1. BERMAN, D.L., On the theory of interpolation, Dokl. Akad. Nauk SSSR 163(3) (1965) (Soviet Math, Dokl. 6 (1965), 945-948).

2. BOJANIC, R., PRASAD, J. and SAXENA, R.B., An upper bound for the rate of convergence of the Hermite-Fejér process on the extended Chebyshev nodes of the second kind, J. Approx. Theory 26 (1979), 195-203.

3. BOJANIC, R., VARMA, A.K. and VÉRTESI, P., Necessary and Sufficient conditions for uniform convergence of Quasi Hermite and extended Hermite-Fejér interpolation, Accepted for publication in Studia Math. Hung.

4. CZIPSZER, J. and FREUD, G., Sur l'approximation d'une foction periodique et de ses dérivées successives par un polynome trigonometrique et par ses derivés successive, Acta Math. 99 (1958), 33-51.

5. EGERVÁRY, E. and TURÁN, P., Notes on interpolation V, Acta Math. Acad. Sci. Hung. 9 (1958), 259-267.

6. ERDÖS, P., On the maximum of the fundamental functions of the ultraspherical polynomials, Ann. of Math. 45 (1944), 335-339.

7. FEJÉR, L., Über Interpolation, Nachr. Gesell. Gött., (1916), 66-91.

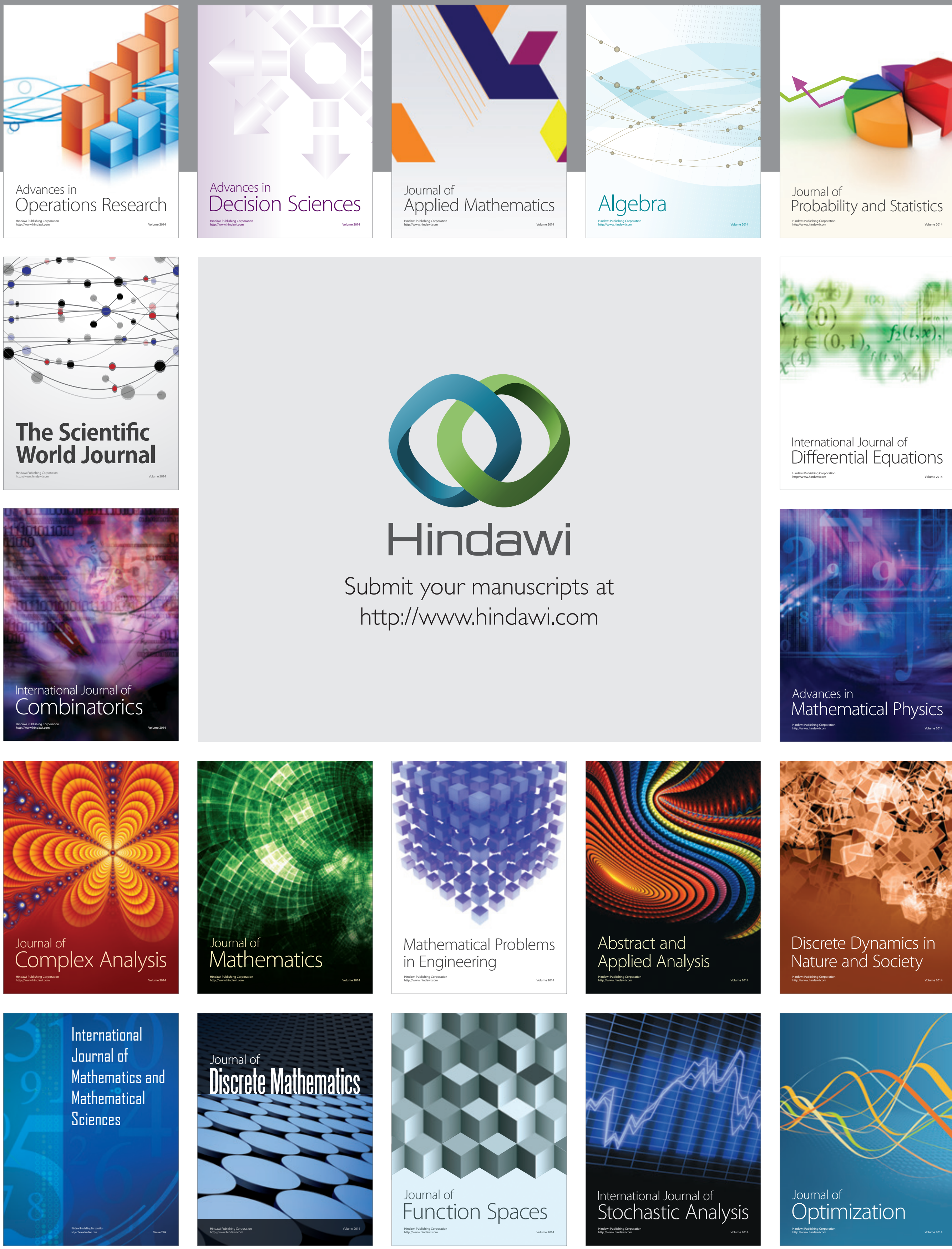
8. PRASAD, J. and SAXENA, R.B., Degree of Convergence of quasi Hermite-Fejér interpolation, Publ. Inst. Math. (N.S.) 19 (33) (1975), 123-130.

9. PRASAD, J. and VARMA, A.K., A study of some interpolatory processes based on the roots of Legendre polynomials, J. Approx. Theory 31(3) (1981), 244-252.

10. RIVLIN, T.J., Introduction to the approximation of functions, Ginn (Blaisdell), Boston, 1969.

11. SZÁSZ, P., On quasi Hermite-Fejér interpolation, Acta Math. Acad. Sci. Hungar. 10 (1959), 413-439.

12. SZEGÖ, G., Orthogonal polynomials, "American Math. Soc. Coll. Publ. No. XXIII," New York, 1959.



Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization