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RESEARCH NOTES

A NOTE ON CONSERVATIVE MEASURES ON SEMIGROUPS

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ABSTRACT. Consider (S,B,μ) the measure space where S is a topological metric semigroup and μ a countably additive bounded Borel measure. Call μ conservative if all right translations t_x: $s \to sx$, $x \in S$ (which are assumed closed mappings) are conservative with respect (S,B,μ) in the ergodic theory sense. It is shown that the semigroup generated by the support of μ is a left group. An extension of this result is obtained for σ-finite μ.

KEY WORDS AND PHRASES. Topological metric semigroups, left groups, conservative translations of the semigroup, Borel measure, r*-invariant measures.
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1. INTRODUCTION.

Throughout we shall be dealing with the measure space (S, B_σ, μ) where S is a topological metric semigroup, B_σ its Borel σ-algebra and μ a non-negative countably additive Borel measure on B_σ. We shall assume that the right translations t_x: $s \to sx$, $x \in S$ (which are measurable) are closed mappings. The support F of μ is defined by $F = \{s \in S;$ every neighborhood of s has positive μ- measure$\}$. We assume that μ is finite on compacta and F^c (=the complement of F) is of μ- measure zero. For $x \in S$, $B \subset S$, $Bx^{-1} = \{s;\ sx \in B\}$. The closure of B will be denoted by $\bar{B}$.

The purpose of this note is to study the effects on S of certain invariant conditions on the t_x's of the "measure-preserving" type especially that of conservativeness and r*-invariance. By abuse of language (interchanging the roles of the t_x's and μ) we shall say that μ is conservative (or recurrent) if every t_x, $x \in S$, is conservative with respect to the measure space (S,B_σ,μ) in the ergodic theory sense , that is, if

$$Bx^{-1} \supset B \Rightarrow \text{(implies)} \ \mu(Bx^{-1} - B) = 0 \quad (1.1)$$
$$\text{for all } B \in B_\sigma, x \in S.$$

Such transformations are familiar in ergodic theory (See [1] and [2]) and are characterized by the property of (infinite) recurrence, i.e.,

$$\forall\ B \in B_\sigma\ ,\ x \in S, \quad \mu\{ B - \bigcap_{m=1}^{\infty} \bigcup_{i\geq m}^{\infty} Bx^{-i} \} = 0 \quad (1.2)$$

Equivalent definitions of conservativeness" would be obtained using closed sets in (1.1) or requiring the non-existence of a closed set K of positive measure such that $K, Kx^{-1}, Kx^{-2}, \ldots\ldots, x \in S,$ is a disjoint sequence. The measure μ is called r*-

invariant if

$$\text{For all } B \in B_\sigma,\ x \in S, \quad \mu(Bx^{-1}) = \mu(B) \tag{1.3}$$

Conditions (1.1) and (1.3), despite their similarity, are really independent. However for bounded μ, trivially r*-invariance implies conservativeness. This fact and the wish to generalize the following result of [3] (Theorem 1, below) lead us to consider the σ-finite case and the effects of conservativeness of μ on S.

2. THE CASE OF FINITE MEASURE.

THEOREM 1. (of [3]). Suppose in (S,B_σ,μ), μ is r*-invariant (not necessarily finite) measure and the t_x, $x \in S$, are closed mappings. Then F is a left group, i.e. F is left simple and right cancellative .

A simple generalization of this theorem (with a new proof) in the case of bounded μ, is the following

THEOREM 2. Suppose in (S, B_σ,μ) μ is bounded conservative. (The t_x's are assumed always closed). Let D be the subsemigroup generated by the support F of μ, i.e., $D = \overline{\bigcup_{n=1} F^n}$, and let $L = \bigcap_{x \in S} Sx$. Then $D \subset L$ and both D and L are left groups.

PROOF. Let us denote by μ^k the k^{th} convolution power of μ. For k = 2,

$$\mu^2(B) = \mu*\mu(B) = \int\mu(By^{-1})\,\mu(dy) = \int\mu(y^{-1}B)\mu(dy) \tag{2.1}$$

Suppose now that $Bx^{-1} \supset B$. Since

$$\mu^2(Bx^{-1} - B) = \int\mu(y^{-1}(Bx^{-1} - B))\,\mu(dy) = \int\mu((y^{-1}B)x^{-1} - y^{-1}B)\mu(dy) = 0 \tag{2.2}$$

we see that all powers μ^k are also conservative. Let $x \in D$ and U any neighborhood of x. Then $\mu^k(U) > 0$ for some k, since D is the semigroup generated by the powers of the support F of μ. By (1.2) applied to μ^k we must have $D \cap Ux^{-j} \neq \emptyset$ for some j and we may assume $j > 2$. Hence $U \cap Dx^{j-1}x \neq \emptyset$ and $U \cap Dx \neq \emptyset$ and a fortiori $U \cap Sx \neq \emptyset$. It follows, since x was arbitrary, that $\overline{Dx} = Dx \supset D$ for all $x \in D$ and hence $Dx = D$, so that D is left simple. Also $\overline{Sx} = Sx \supset D$ for all $x \in S$ and $L = \bigcap_{x\in S} Sx$ is left simple since it is the minimal left ideal of S. Incidentally, also $L' = \bigcap_{x \in S} Dx$ is left simple semigroup and $L \supset L' \supset D$.

Next, we show that there is an idempotent element in D. Let $a \in D$. Since D is left simple $aa^{-1} \neq \emptyset$. We consider two cases. Case 1: Interior$(aa^{-1}) = \emptyset$. Then it follows that Frontier(aa^{-1}) is compact by a result of Morita and Hanai (Proc. Jap. Acad. 32 (1956), p. 10-14) (cf. also [3] ,p. 318), and being a compact semigroup, it contains an idempotent. Case 2: Suppose Interior$(aa^{-1}) \neq \emptyset$. Then for some k, $\mu^k(aa^{-1}) > 0$. This implies by (1.2) applied to μ^k, that $\mu^k(aa^{-1} - \bigcup_{i=1}^{\infty} (aa^{-1})a^{-i})) = 0$. It follows that there is $v \in (aa^{-1})a^{-j} \cap aa^{-1}$ for some j, so that $va^j a = a$ or $vaa^{j-1}a = a$ or $a^{j+1} = a$ and hence the powers of a form a finite semigroup which must contain an idempotent element. Therefore, D is a left group, since an alternate characterization of left group is that it be left simple and contain an idempotent element.

3. THE CASE OF INFINITE MEASURE.

One may wonder what is an appropriate condition under which an infinite r*-invariant μ becomes conservative. One such condition (admittedly not very manageable) is the non-existence of an unbounded (in measure) G_δ set G such that $Gx^{-1} \supset G$ for some x and $\mu(Gx^{-1} - G) > 0$, that is, for all unbounded G_δ sets G such that $Gx^{-1} \supset G$ for some x, we have $\mu(Gx^{-1} - G) = 0$. Such a condition plus r*-invariance imply (1.1).

In the infinite case we have obtained only partial results summarized in the following theorem.

THEOREM 3. Suppose in (S, B_σ, μ) μ is an (infinite) conservative measure and the right translations t_x, $x \in S$, are closed. Let F denote the support of μ. Let $D = \overline{\bigcup_{i=1} F^i}$ and $E = \bigcap_{k=1}^{\infty} \overline{\bigcup_{i\geq k} F^{i+1}}$. Then,

(i) $F \subset (\bigcap_{x\in E} Ex) \cap (\bigcap_{x\in S} Sx)$ and $F \subset E \subset D$, E being a closed subsemigroup.

(ii) If μ is σ-finite and S is separable, then $E = D$ = a left group.

(iii) If $F = S$ or F is a subsemigroup, then F is a left group.

REMARK. A condition that makes F a subsemigroup is the non-contractiveness of μ i.e., for closed sets B, $\mu(B) > 0$ implies $\mu(Bx) > 0$, $x \in F$.

PROOF. (i): Let k be given. For any neighborhood U of a point $f \in F$ there is an $i > k$, depending on U, such that by (1.2) $Uf^{-i} \cap F \neq \emptyset$ and hence $U \cap Ff^i \neq \emptyset$ and $U \cap F^{i+1} \neq \emptyset$. Hence $F \subset \overline{\bigcup_{i\geq k} F^{i+1}}$ for every $k \geq 1$. Let now U be as above and $x \in E$. Using (1.2), since $Ux^{-i} \cap F \neq \emptyset$ implies $U \cap Fx^{i-1}x \neq \emptyset$ (we may take $i > 2$) and Fx^{i-1} is in E (E being an ideal in D), we have $F \subset \bigcap_{x\in E} Ex$.

(ii): The separability of S and σ-finiteness of μ imply that the functions $\mu(Bx^{-1})$ and $\mu(x^{-1}B)$, $B \in B_\sigma$, are measurable and also the validity of (2.1) and (2.2) (cf. [6] and [7]). Hence, that the convolution powers of μ are again conservative. Then, as in Theorem 2, D is a left group. (the conservativeness of μ^k is needed for left simplicity and to produce an idempotent element in D,as in Theorem 2).Observe that E being an ideal in D must equal to D since D is now left simple.

(iii): We can prove F to be left simple either by observing that for x in F, $Fx \subset Fxxx^{-1} \subset Fxx^{-1}$ and hence $\mu(Fxx^{-1} - Fx) = 0$ by (1.1), so that $\mu(F - Fx) = 0$ since $F \subset Fxx^{-1}$ and $F - Fx$, being open, is empty, or, we may use (1.2) and the argument in Theorem 2 showing that $F \subset \overline{Fx} = Fx$. To produce an idempotent we use the argument on the Interior(aa^{-1}), $a \in F$, and μ^k with $k = 1$, as in Theorem 2. (Here we don't need the convolution powers μ^k for $k > 1$ to be conservative).

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