

**ON A NONLINEAR DEGENERATE EVOLUTION EQUATION
 WITH STRONG DAMPING**

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ABSTRACT. In this paper we consider the nonlinear degenerate evolution equation with strong damping,

$$(*) \quad \begin{cases} K(x, t)u_t - \Delta u - \Delta u_t + F(u) = 0 & \text{in } Q = \Omega \times]0, T[\\ u(x, 0) = u_0, (Ku')(x, 0) = 0 & \text{in } \Omega \\ u(x, t) = 0 & \text{on } \Sigma = \Gamma \times]0, T[\end{cases}$$

where K is a function with $K(x, t) \geq 0$, $K(x, 0) = 0$ and F is a continuous real function satisfying

$$(**) \quad sF(s) \geq 0, \quad \text{for all } s \in \mathbb{R},$$

Ω is a bounded domain of \mathbb{R}^n , with smooth boundary Γ . We prove the existence of a global weak solution for (*).

KEY WORDS AND PHRASES. Weak solutions, evolution equation with damping.

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1. INTRODUCTION.

In this work we study the existence of global weak solutions for the degenerate problem

$$(1.1) \quad \begin{cases} K(x, t)u'' - \Delta u - \Delta u' + F(u) = 0 \\ u(0) = u_0 \\ (Ku')(0) = 0 \\ u = 0 \end{cases} \quad \text{in } \Sigma$$

in the cylinder $Q = \Omega \times]0, T[$ where Ω is a bounded domain in \mathbb{R}^n with smooth boundary, $T > 0$ is an arbitrary real number, Σ is a lateral boundary of Q , F is a continuous real function such that $sF(s) \geq 0$, for all $s \in \mathbb{R}$, $K : Q \rightarrow \mathbb{R}$ is a function such that $K(x, t) \geq 0$, $(x, t) \in Q$, $K(x, 0) = 0$, Δ is the Laplace operator and $u' = \frac{\partial u}{\partial x}$.

Equation (1.1) is a nonlinear perturbation of the wave equation. For $n = 1$ or $n = 2$, (1.1) governs the motion of a linear Kelvin solid (a bar if $n = 1$ and a plate if $n = 2$) subject to no nonlinear elastic constraints, where $K(x, t)$ is a mass density.

Problem (1.1) with $K(x, t) = 1$ without the term $-\Delta u'$ was studied by Strauss [1]. He proves the existence of global weak solutions and the asymptotic behavior as t approaches to infinity. The global weak solutions for the equation

$$K_1(x, t)u'' + K_2(x, t)u' - \Delta u + F(u) = 0 \tag{1.2}$$

with $K_1(x, t) \geq 0, K_1(x, 0) \geq \alpha > 0$ and $K_2(x, t) \geq \beta > 0$ was studied by Maciel [2].

Problem (1.2) was also studied by Mello [3] for $F \in C^1(\mathbb{R}), F(0) = 0, \int_0^1 F(\xi)d\xi \geq 0, F'$ dominated by $|s|^p, p > 0, K_2$ independent of t non-zero initial data.

In [4] and [5], Larkin studied problem (1.2) with $F(u) = |u|^p u$ and $F(u) = |u'|^p u', p > 0$, respectively. In both cases the initial data are zero.

Problem (1.1) with $K(x, t) = 1$ was studied by Ang and Dinh [6] with $F \in C^1(\mathbb{R}), F(0) = 0$ and $F' \geq -C$ with $C > 0$ "small." They proved the existence of global weak solutions and the asymptotic behavior when t approaches to infinity.

We denote by $(\cdot, \cdot), |\cdot|, ((\cdot, \cdot)), \|\cdot\|$ the inner and norm of $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively, and $a(u, v) = \sum_{i=1}^n \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx$ represents Dirichlet's form in $H_0^1(\Omega)$.

2. ASSUMPTIONS AND MAIN RESULTS.

We consider the following hypothesis:

- (H.1) $F : \mathbb{R} \rightarrow \mathbb{R}$ is continuous with $sF(s) \geq 0, \forall s \in \mathbb{R}$;
- (H.2) $K \in C^1([0, T] : L^\infty(\Omega))$ with $K(x, t) \geq 0, (x, t) \in Q$ and $K(x, 0) = 0$
- (H.3) $\left| \frac{\partial K}{\partial x} \right| \leq \delta + C(\delta)K, \forall \delta > 0$ where $C(\delta)$ is a positive constant.

Then we have the following result:

THEOREM 1. Under hypothesis (H.1)-(H.3) if $G(s) = \int_0^s F(\xi)d\xi$ and $u_0 \in H_0^1(\Omega), G(u_0) \in L^1(\Omega)$ then there exists a function $u : [0, T] \rightarrow L^2(\Omega)$ such that:

$$u \in L^\infty(0, T : H_0^1(\Omega)) \tag{2.1}$$

$$u' \in L^\infty(0, T : H_0^1(\Omega)) \tag{2.2}$$

$$\sqrt{K(x, t)}u' \in L^\infty(0, T : L^2(\Omega)) \tag{2.3}$$

$$K'(x, t)u' \in L^2(0, T : H_0^1(\Omega)) \tag{2.4}$$

$$\frac{d}{dt}(Ku', v) - (K'u', v) + a(u, v) + a(u', v) + (F(u), v) = 0 \text{ in } \mathcal{D}(0, T), \forall v \in H_0^1(\Omega) \tag{2.5}$$

$$u(0) = u_0 \tag{2.6}$$

$$(Ku')(0) = 0 \tag{2.7}$$

We divide the proof in two parts:

- i) We consider F Lipschitzian and derivable except on a finite number of points with $sF(s) \geq 0, \forall s \in \mathbb{R}$.

- ii) We consider F continuous with $F(s) \geq 0, \forall s \in \mathbb{R}$ and approximate F by a sequence $(F_\eta)_{\eta \in \mathbb{N}}$ F_η Lipschitzian and derivable except on a finite number of points with $sF_\eta(s) \geq 0, \forall s \in \mathbb{R}, \forall \eta \in \mathbb{N}$, with $F_\eta \rightarrow F$ uniformly on bounded sets of \mathbb{R} .

2.1 LIPSCHITZIAN CASE

We have the following result:

THEOREM 2. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be such that $sF(s) \geq 0$, Lipschitzian and derivable except on a finite number of points. Let be $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ with $G(u_0) \in L^1(\Omega)$, where $G(s) = \int_0^s F(\xi)d\xi$.

Then there exists a unique function $u : Q \rightarrow \mathbb{R}$ satisfying:

$$u \in L^\infty(0, T; H_0^1(\Omega)) \tag{2.8}$$

$$u' \in L^\infty(0, T; H_0^1(\Omega)) \tag{2.9}$$

$$u'' \in L^2(0, T; H_0^1(\Omega)) \tag{2.10}$$

$$K(x, t)u'' - \Delta u - \Delta u' + F(u) = 0 \text{ in } L^2(0, T; H^{-1}(\Omega)) \tag{2.11}$$

$$u(0) = u_0, \quad u'(0) = 0. \tag{2.12}$$

PROOF. Let $(w_\nu)_{\nu \in \mathbb{N}}$ be a basis of $H_0^1(\Omega) \cap H^2(\Omega)$ and $V_m = [w_1, \dots, w_m]$ the subspace generated by the m first vectors of $(w_\nu)_{\nu \in \mathbb{N}}$.

2.1.1 APPROXIMATION PERTURBED PROBLEM

Fix $\varepsilon > 0$ and for each $m \in \mathbb{N}$ consider a function of the form

$$u_{\varepsilon m}(t) = \sum_{j=1}^m g_{j\varepsilon m}(t)w_j$$

such that $u_{\varepsilon m}(t)$ is a solution of the problem:

$$((K + \varepsilon)u_{\varepsilon m}'', w) + a(u_{\varepsilon m}, w) + a(u_{\varepsilon m}', w) + (F(u_{\varepsilon m}), w) = 0, \quad \forall w \in V_m \tag{2.13}$$

$$u_{\varepsilon m}(0) = u_{0m} \rightarrow u_0 \text{ strongly in } H_0^1(\Omega) \cap H^2(\Omega) \tag{2.14}$$

$$u_{\varepsilon m}'(0) = 0 \tag{2.15}$$

By Caratheodory's theorem, $u_{\varepsilon m}(t)$ exists on $[0, T_{\varepsilon m}[$, $T_{\varepsilon m} < T$. The a priori estimates will allow us to extend $u_{\varepsilon m}(t)$ to whole interval $[0, T]$.

2.1.2 A PRIORI ESTIMATES

I) Consider $w = u_{\varepsilon m}'(t)$ in (2.13). We obtain

$$\frac{1}{2} \frac{d}{dt} \left[(K, u_{\varepsilon m}'^2) + \varepsilon |u_{\varepsilon m}'|^2 + \|u_{\varepsilon m}\|^2 + 2 \int_{\Omega} G(u_{\varepsilon m})dx \right] + \|u_{\varepsilon m}'\|^2 = \frac{1}{2} \left[\frac{\partial K}{\partial t}, u_{\varepsilon m}'^2 \right]$$

Integrating from 0 to $t \leq T_{\varepsilon m}$ and using (H.3) we get:

$$\begin{aligned} & (K, u_{\varepsilon m}'^2) + \varepsilon |u_{\varepsilon m}'|^2 + \|u_{\varepsilon m}\|^2 + 2 \int_{\Omega} G(u_{\varepsilon m})dx + 2 \int_0^t \|u_{\varepsilon m}'\|^2 ds \\ & \leq \|u_{0m}\|^2 + 2 \int_{\Omega} G(u_{0m})dx + \int_0^t [\delta |u_{\varepsilon m}'|^2 + C(\delta)(K, u_{\varepsilon m}'^2)] ds \end{aligned}$$

By (2.14) and because $G(u_0) \in L'(\Omega)$ we have:

$$\int_{\Omega} G(u_{0m})dx \rightarrow \int_{\Omega} G(u_0)dx \tag{2.16}$$

By (2.14)-(2.16) and Gronwall's inequality, it follows that:

$$(K, u_{\epsilon m}'^2) + \epsilon |u_{\epsilon m}'|^2 + \|u_{\epsilon m}\|^2 + 2 \int_{\Omega} G(u_{\epsilon m})dx + (2 - \tilde{C}\delta) \int_0^t \|u_{\epsilon m}'\|^2 ds \leq M$$

where M is a positive constant independent of $\epsilon, m, t, \tilde{C}$ is a positive constant such that $|v|^2 \leq \tilde{C} \|v\|^2$ and

$\delta < \min \left\{ 2, \frac{2}{\tilde{C}} \right\}$. Thus

$$\left(K^{\frac{1}{2}} u_{\epsilon m}' \right) \text{ is bounded in } L^{\infty}(0, T; L^2(\Omega)) \tag{2.17}$$

$$(u_{\epsilon m}) \text{ is bounded in } L^{\infty}(0, T; H_0^1(\Omega)) \tag{2.18}$$

$$(u_{\epsilon m}') \text{ is bounded in } L^2(0, T; H_0^1(\Omega)) \tag{2.19}$$

$$(\sqrt{\epsilon} u_{\epsilon m}') \text{ is bounded in } L^{\infty}(0, T; L^2(\Omega)) \tag{2.20}$$

II) Since F is Lipschitzian and derivable except on a finite number of points of \mathbb{R} , we can differentiate with respect to t to obtain

$$\left[\frac{\partial K}{\partial t} u_{\epsilon m}'', w \right] + (K u_{\epsilon m}''', w) + \epsilon (u_{\epsilon m}''', w) + a(u_{\epsilon m}', w) + a(u_{\epsilon m}'', w) + (F'(u_{\epsilon m}) u_{\epsilon m}', w) = 0 \tag{2.21}$$

Taking $w = u_{\epsilon m}''(t)$ in (2.21), we get

$$\frac{d}{dt} \left[(K, u_{\epsilon m}''^2) + \epsilon |u_{\epsilon m}''|^2 + \|u_{\epsilon m}'\|^2 \right] + 2 \|u_{\epsilon m}''\|^2 + \left[\frac{\partial K}{\partial t}, u_{\epsilon m}''^2 \right] + 2(F'(u_{\epsilon m}), u_{\epsilon m}'') = 0 \tag{2.22}$$

But

$$2(F'(u_{\epsilon m}) u_{\epsilon m}', u_{\epsilon m}'') \leq 2 |F'(u_{\epsilon m})| \|u_{\epsilon m}'\| |u_{\epsilon m}''| \leq 2\beta |u_{\epsilon m}'| |u_{\epsilon m}''| \tag{2.23}$$

where β is a positive constant.

Integrating (2.22) from 0 to t and using (2.14)-(2.15), (2.23) and (H.3), it follows that

$$\begin{aligned} & (K, u_{\epsilon m}''^2 + \epsilon |u_{\epsilon m}''|^2 + \|u_{\epsilon m}'\|^2) + (2 - \delta) \int_0^t \|u_{\epsilon m}''\|^2 ds \\ & \leq \epsilon |u_{\epsilon m}''(0)|^2 + C_1 \int_0^t \left[\|u_{\epsilon m}'\|^2 + (K, u_{\epsilon m}''^2) \right] ds \end{aligned} \tag{2.24}$$

where C_1 is a positive constant.

Now, we are going to estimate the term $\epsilon |u_{\epsilon m}''(0)|^2$. Consider $t = 0$ in (2.13), and $w = u_{\epsilon m}''(0)$. Then we get

$$\epsilon |u_{\epsilon m}''(0)| \leq |\Delta u_{0m}| + |F(u_{0m})| \leq C \tag{2.25}$$

where C is a positive constant independent of ϵ, m and t .

By (2.24), (2.25) and Gronwall's inequality, there exists a positive constant M_1 , independent of ϵ, m and t , such that:

$$(K, u_{\epsilon m}''^2) + \epsilon |u_{\epsilon m}'|^2 + \|u_{\epsilon m}'\|^2 + (2 - \delta) \int_0^t \|u_{\epsilon m}''\|^2 ds \leq M_1$$

So,

$$\left(K^{\frac{1}{2}} u_{\epsilon m}'\right) \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \tag{2.26}$$

$$(\sqrt{\epsilon} u_{\epsilon m}'') \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \tag{2.27}$$

$$(u_{\epsilon m}') \text{ is bounded in } L^\infty(0, T; H_0^1(\Omega)) \tag{2.28}$$

$$(u_{\epsilon m}'') \text{ is bounded in } L^2(0, T; H_0^1(\Omega)) \tag{2.29}$$

2.1.3 Limits of the Approximated Solutions

From the estimates (2.17)-(2.20) and (2.26)-(2.29), there exists a subsequence of $(u_{\epsilon m})$, which we still denote by $(u_{\epsilon m})$, such that:

$$u_{\epsilon m} \rightarrow u \text{ weakly - star in } L^\infty(0, T; H_0^1(\Omega)) \tag{2.30}$$

$$u_{\epsilon m}' \rightarrow u' \text{ weakly - star in } L^\infty(0, T; H_0^1(\Omega)) \tag{2.31}$$

$$u_{\epsilon m}' \rightarrow u' \text{ weakly in } L^2(0, T; H_0^1(\Omega)) \tag{2.32}$$

$$\sqrt{\epsilon} u_{\epsilon m}'' \rightarrow 0 \text{ weakly - star in } L^\infty(0, T; L^2(\Omega)) \tag{2.33}$$

$$Ku_{\epsilon m}'' \rightarrow Ku'' \text{ weakly - star in } L^\infty(0, T; L^2(\Omega)) \tag{2.34}$$

By (2.18), (2.19) and compactness arguments we conclude that there exists a subsequence of $(u_{\epsilon m})$, which we still denote by $(u_{\epsilon m})$, such that:

$$u_{\epsilon m} \rightarrow u \text{ strongly in } L^2(0, t; L^2(\Omega)) = L^2(Q). \tag{2.35}$$

Thus,

$$u_{\epsilon m} \rightarrow u \text{ almost everywhere in } Q.$$

whence, by (H.1) we have

$$F(u_{\epsilon m}) \rightarrow F(u) \text{ almost everywhere in } Q \tag{2.36}$$

Since $K \in C^1([0, T; L^\infty(\Omega)])$, using (2.32) we obtain

$$(Ku_{\epsilon m}'') \text{ is bounded in } L^2(Q) \tag{2.37}$$

Then,

$$Ku_{\epsilon m}'' \rightarrow Ku'' \text{ weakly in } L^2(Q) \tag{2.38}$$

Taking $w = u_{\epsilon m}(t)$ in (2.13), integrating from 0 to t and using (2.18), (2.19) and (2.37), we get

$$\int_Q F(u_{\epsilon m}(t)) u_{\epsilon m}(t) dx dt \leq C \tag{2.39}$$

where C is a positive constant.

By (2.36), (2.39) and Strauss's theorem (see Strauss [1]) it follows that

$$F(u_{\epsilon m}) \rightarrow F(u) \text{ weakly in } L^1(Q) \tag{2.40}$$

Multiplying (2.13) by $\theta \in L^2(0, T)$, integrating from 0 to t and taking the limit as $m \rightarrow \infty$ and $\varepsilon \rightarrow 0$, we obtain, by (2.30)-(2.34), (2.38) and (2.40):

$$\left(\int_0^T Ku''\theta dt, \omega \right) + \left(\int_0^T -\Delta u\theta dt, w \right) + \left(\int_0^T -\Delta u'\theta dt, w \right) + \left(\int_0^T F(u)\theta dt, w \right) = 0, \quad \forall w \in V_m.$$

Since the V_m is dense in $H_0^1(\omega)$, the above equation is true for all $w \in H_0^1(\Omega)$ and the proof of (2.11) is complete.

The initial conditions (2.12) are obtained from (2.30)-(2.32).

The uniqueness is trivial because F is Lipschitzian.

3. PROOF OF THEOREM 1

We first approximate u_0 by a sequence of bounded functions $(u_{0j})_{j \in \mathbb{N}}$ in $H_0^1(\Omega)$. In fact, let's consider

$$\beta_j(s) = \begin{cases} s & \text{if } |s| \leq j \\ j & \text{if } s > j \\ -j & \text{if } s < -j \end{cases}$$

it follows by Kinderlher-Stampacchia [8] that $\beta_j(u_0) = u_{0j} \in H_0^1(\Omega), \forall j \in \mathbb{N}, u_{0j} \rightarrow u_0$ strongly in $H_0^1(\Omega)$ and $\|u_{0j}\| \leq \|u_0\|$.

Let $(F_\eta)_{\eta \in \mathbb{N}}$ be a sequence of functions defined by:

$$F_\eta(s) = \begin{cases} (-\eta) \left[G\left(s - \frac{1}{\eta}\right) - G(s) \right] & \text{if } -\eta \leq s \leq -\frac{1}{\eta} \\ \eta \left[G\left(s + \frac{1}{\eta}\right) - G(s) \right] & \text{if } \frac{1}{\eta} \leq s \leq \eta \\ \text{linear by parts} & \text{on } -\frac{1}{\eta} \leq s \leq \frac{1}{\eta} \text{ with } F_\eta(0) = 0 \\ \text{appropriated constants} & \text{for } |s| \geq \eta \end{cases}$$

where

$$G(s) = \int_0^s F(\xi) d\xi.$$

It follows, by Strauss [1], Cooper-Medeiros [7] that F_η is Lipschitzian, for each $\eta \in \mathbb{N}, sF_\eta(s) \geq 0$ and $F_\eta \rightarrow F$ uniformly on the bounded sets of \mathbb{R} . If we consider $G_\eta(s) = \int_0^s F_\eta(\xi) d\xi$ we get, $G_\eta(0) = 0$ and $sG_\eta(s) \geq 0, \forall s \in \mathbb{R}, \forall \eta \in \mathbb{N}$.

Let $\phi_{\mu j} \in \mathcal{D}(\Omega)$ such that

$$\phi_{\mu j} \rightarrow u_{0j} \text{ strongly in } H_0^1(\Omega) \text{ as } \mu \rightarrow \infty \tag{3.1}$$

It follows by Theorem 2 that there exists a unique function $u_{\mu j \eta}$ satisfying the conditions:

$$u_{\mu_j\eta} \in L^\infty(0, T; H_0^1(\Omega)) \tag{3.2}$$

$$u'_{\mu_j\eta} \in L^\infty(0, T; H_0^1(\Omega)) \tag{3.3}$$

$$u''_{\mu_j\eta} \in L^2(0, T; H_0^1(\Omega)) \tag{3.4}$$

$$Ku'_{\mu_j\eta} - \Delta u_{\mu_j\eta} - \Delta u'_{\mu_j\eta} + F(u_{\mu_j\eta}) = 0 \quad \text{in } L^2(0, T; H^{-1}(\Omega)) \tag{3.5}$$

$$u_{\mu_j\eta}(0) = \phi_{\mu_j}, \quad u'_{\mu_j\eta}(0) = 0 \tag{3.6}$$

We now prove that $u_{\mu_j\eta}$ converges to u and u is the solution of Theorem 1. Taking the inner product of (3.5) by $u'_{\mu_j\eta}$ and integrating from 0 to $t \leq T$, we have:

$$\begin{aligned} & (K, u_{\mu_j\eta}'^2) + \|u_{\mu_j\eta}\|^2 + 2 \int_{\Omega} G_{\eta}(u_{\mu_j\eta}) dx + 2 \int_0^t \|u'_{\mu_j\eta}\|^2 ds \\ & \leq \|\phi_{\mu_j}\|^2 + 2 \int_{\Omega} G_{\eta}(\phi_{\mu_j}) dx + \int_0^t [\delta |u'_{\mu_j\eta}|^2 + C(\delta)(K, u_{\mu_j\eta}'^2)] ds. \end{aligned} \tag{3.7}$$

Since u_{0_j} is bounded in Ω , fixing j , we obtain:

$$F_{\eta}(u_{0_j}(x)) \rightarrow F(u_{0_j}(x)) \quad \text{uniformly in } \Omega \text{ as } \eta \rightarrow \infty, \tag{3.8}$$

$$\int_{\Omega} G_{\eta}(\phi_{\mu_j}) dx \rightarrow \int_{\Omega} G_{\eta}(u_{0_j}) dx \quad \text{if } \mu \rightarrow +\infty. \tag{3.9}$$

and

$$(G_{\eta}(u_{0_j}(x)) \rightarrow G(u_{0_j}(x)) \quad \text{uniformly in } \Omega \text{ as } \eta \rightarrow \infty. \tag{3.10}$$

Whence, there exists a subsequence $(G_{\eta_j})_{j \in \mathbb{N}}$ of $(G_{\eta})_{\eta \in \mathbb{N}}$, which we still denote by $(G_j)_{j \in \mathbb{N}}$, such that

$$\int_{\Omega} |G_j(u_{0_j}) - G(u_{0_j})| dx \rightarrow 0 \quad \text{if } j \rightarrow \infty. \tag{3.11}$$

Moreover, $G(u_{0_j}) \rightarrow G(u_0)$ a.e. in Ω and $G(u_{0_j}) \leq G(u_0)$. Since $G(u_0) \in L^1(\Omega)$, by the Lebesgue's dominated convergence theorem we get

$$\int_{\Omega} |G(u_{0_j}) - G(u_0)| dx \rightarrow 0 \quad \text{as } j \rightarrow \infty, \tag{3.12}$$

Thus, by (3.11) and (3.12), it follows that

$$\int_{\Omega} G_j(u_{0_j}) dx \rightarrow \int_{\Omega} G(u_0) dx \quad \text{as } j \rightarrow \infty \tag{3.13}$$

By (3.7), (3.9), (3.13) and Gronwall's inequality, we have

$$(K, u_{\mu_j}^2) + \|u_{\mu_j}\|^2 + 2 \int_{\Omega} G_j(u_{\mu_j}) dx + (2 - C\delta) \int_0^t \|u'_{\mu_j}\|^2 dx \leq C, \tag{3.14}$$

where C is a positive constant independent of μ , j and t .

Then, there exists a subsequence of $(u_{\mu_j})_{\mu \in \mathbb{N}}$, which we denote by $(u_{\mu})_{\mu \in \mathbb{N}}$, and functions u_j and u such that

$$\begin{cases} K^{1/2}u'_{\mu j} \rightarrow K^{1/2}u'_j & \text{weakly - star in } L^\infty(0, T; L^2(\Omega)) \\ u_{\mu j} \rightarrow u_j & \text{weakly - star in } L^\infty(0, T; H_0^1(\Omega)) \\ u'_{\mu j} \rightarrow u'_j & \text{weakly in } L^2(0, T; H_0^1(\Omega)) \end{cases} \tag{3.15}$$

as $\mu \rightarrow \infty$, and

$$\begin{cases} K^{1/2}u'_j \rightarrow K^{1/2}u' & \text{weakly - star in } L^\infty(0, t; L^2(\Omega)) \\ u_j \rightarrow u & \text{weakly - star } L^2(0, T; H_1^1(\Omega)) \\ u'_j \rightarrow u' & \text{weakly in } L^2(0, T; H_0^1(\Omega)) \end{cases} \tag{3.16}$$

as $j \rightarrow \infty$.

Moreover, by (H.2) and $K^{1/2}u'_{\mu j} \in L^\infty(0, T; L^2(\Omega))$ it follows that:

$$Ku'_{\mu j} \in L^\infty(0, T; L^2(\Omega)) \tag{3.17}$$

and

$$Ku'_{\mu j} \rightarrow Ku'_j \text{ weakly - star in } L^\infty(0, T; L^2(\Omega)) \tag{3.18}$$

as $\mu \rightarrow \infty$, and

$$Ku'_j \rightarrow Ku' \text{ weakly - star in } L^\infty(0, T; L^2(\Omega)) \tag{3.19}$$

as $j \rightarrow \infty$.

By (H.2), (H.3), (3.3) and (3.4) we get

$$(Ku')' \in L^2(Q). \tag{3.20}$$

So, by (3.18) and (3.19) we have that $Ku'_{\mu j}$ is weakly continuous of $[0, T]$ in $L^2(\Omega)$. Moreover, $(Ku'_{\mu j})(T)$ is bounded in $L^2(\Omega)$.

Multiplying (3.5) by $u_{\mu j}(t)$ and integrating from 0 to T , we obtain

$$\begin{aligned} \int_0^T (F_j(u_{\mu j}), u_{\mu j}) dt &\leq \int_0^T \|u_{\mu j}\|^2 dt + \int_0^T \left| \left(\frac{\partial K}{\partial t} u'_{\mu j}, u_{\mu j} \right) \right| dt + \int_0^T \left| \left(\frac{\partial K}{\partial t} u'_{\mu j}, u'_{\mu j} \right) \right| dt \\ &+ \int_0^T |a(u'_{\mu j}, u_{\mu j})| dt + |(Ku'_{\mu j})(T), u_{\mu j}(T)| + |(Ku'_{\mu j})(0), u_{\mu j}(0)|. \end{aligned} \tag{3.21}$$

Using (H.2), (H.3) and a priori estimates, it follows that

$$\int_Q F_j(u_{\mu j}) u_{\mu j} dx dt \leq C, \tag{3.22}$$

C positive constant independent of μ, j and t .

Just as in Theorem 1, we prove that:

$$F_j(u_{\mu j}) \rightarrow F(u_j) \text{ a.e. in } Q \text{ as } \mu \rightarrow \infty \tag{3.23}$$

whence by (3.22), (3.23) and Strauss's theorem (see Strauss [1]), we have

$$F_j(u_{\mu j}) \rightarrow F_j(\mu_j) \text{ weakly in } L^1(Q) \text{ as } \mu \rightarrow \infty. \tag{3.24}$$

Also, by (H.3) and (3.14) it follows that

$$(K'u'_{\mu j}) \text{ is bounded in } L^2(Q). \tag{3.25}$$

So

$$K'u'_{\mu j} \rightarrow K'u'_j \text{ weakly in } L^2(Q) \text{ as } j \rightarrow \infty \tag{3.26}$$

and

$$K'u'_j \rightarrow K'u' \text{ weakly in } L^2(Q) \text{ as } j \rightarrow \infty. \tag{3.27}$$

Multiplying (3.5) by $w = v\theta$ with $v \in H_0^1(\Omega)$ and $\theta \in \mathcal{D}(0, T)$, integrating from 0 to T , taking the limit as $\mu \rightarrow \infty$, and using (3.15), (3.16), (3.18), (3.24) and (3.26) we get

$$\frac{d}{dt}(Ku'_j, v) - (K'u_j, v) + a(u_j, v) + a(u'_j, v) + (F_j(u_j), v) = 0 \quad \forall v \in H_0^1(\Omega) \text{ in } \mathcal{D}'(0, T). \tag{3.28}$$

$$u_j(0) = u_0 \text{ and } (Ku'_j)(0) = 0. \tag{3.29}$$

Moreover, by (3.24), it follows that:

$$F_j(u_j) \rightarrow F(u) \text{ weakly in } L^1(Q). \tag{3.30}$$

Taking the limit in (3.28) as $j \rightarrow \infty$ and using (3.16), (3.19), (3.27) and (3.30) we prove (2.1)-(2.5) in theorem 1.

It's not difficult to check that $u(0) = u_0$ and $(Ku')(0) = 0$.

REMARK. Replacing (H.2) by (H.2)' $K \in C^1([0, T]: L^\infty(\Omega))$ with $K(x, 0) \geq \alpha > 0$,

$$K(x, t) \geq 0, \quad (x, t) \in Q.$$

we get with the same arguments

THEOREM 3. Under hypotheses (H.1), (H.2)', (H.3) if $G(s) = \int_0^s F(\xi)d\xi$ and $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $G(u_0) \in L^1(\Omega)$, then there exists a function $u : [0, T] \rightarrow L^2(\Omega)$ such that

$$u \in L^\infty(0, T; H_0^1(\Omega))$$

$$u' \in L^2(0, T; H_0^1(\Omega))$$

$$\sqrt{K}u' \in L^\infty(0, T; L^2(\Omega))$$

$$K'u' \in L^2(0, T; H_0^1(\Omega))$$

$$Ku'' - \Delta u - \Delta u' + F(u) = 0 \text{ in the weak sense in } Q$$

$$u(0) = u_0$$

$$u'(0) = u_1$$

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