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ON POLYNOMIAL EP_r MATRICES

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ABSTRACT. This paper gives a characterization of EP_r-λ-matrices. Necessary and sufficient conditions are determined for (i) the Moore-Penrose inverse of an EP_r-λ-matrix to be an EP_r-λ-matrix and (ii) Moore-Penrose inverse of the product of EP_r-λ-matrices to be an EP_r-λ-matrix. Further, a condition for the generalized inverse of the product of λ-matrices to be a λ-matrix is determined.

KEY WORDS AND PHRASES: EP_r-λ-matrices, generalized inverse of a matrix.
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1. INTRODUCTION

Let $F[\lambda]^{mxn}$ be the set of all mxn matrices whose elements are polynomials in λ over an arbitrary field F with an involutary automorphism α : $a \leftrightarrow \bar{a}$ for $a \in F$. The elements of $F[\lambda]^{mxn}$ are called λ-matrices. For $A(\lambda) = (a_{ij}(\lambda)) \in F[\lambda]^{mxn}$, $A^*(\lambda) = (\bar{a}_{ji}(\lambda))$. Let $F(\lambda)^{mxn}$ be the set of all mxn matrices whose elements are rational functions of the form $f(\lambda)/g(\lambda)$ where $f(\lambda)$, $g(\lambda) \neq 0$ are polynomials in λ. For simplicity, let us denote $A(\lambda)$ by A itself.

The rank of $A \in F[\lambda]^{mxn}$ is defined to be the order of its largest minor that is not equal to the zero polynomial ([2]p.259). $A \in F[\lambda]^{nxn}$ is said to be an unimodular λ-matrix (or) invertible in $F[\lambda]^{nxn}$ if the determinant of $A(\lambda)$, that is, det $A(\lambda)$ is a nonzero constant. $A \in F[\lambda]^{nxn}$ is said to be a regular λ-matrix if and only if it is of rank n ([2]p.259), that is, if and only if the kernel of A contains only the zero element. $A \in F[\lambda]^{nxn}$ is said to be EP_r over the field $F(\lambda)$ if rk (A) = r and $R(A) = R(A^*)$ where R(A) and rk (A) denote the range space of A and rank of A respectively [4]. We have {unimodular λ-matrices} $\subsetneq$ {regular λ-matrices} $\subsetneq$ {EP-λ-matrices}.

Throughout this paper, let $A \in F_r^{nxn}[\lambda]$. Let 1 be identity element of F. The Moore-Penrose inverse of A, denoted by A^+ is the unique solution of the following set of equations:

$$AXA=A \ (1.1); \quad XAX=X \ (1.2); \quad (AX)^*=AX \ (1.3); \quad (XA)^*=XA \ (1.4)$$

A^+ exists and $A^+ \in F[\lambda]^{nxn}$ if and only if $rk\,(AA^*) = rk\,(A^*A) = rk\,(A)$ [7]. When A^+ exists, A is EP_r over $F(\lambda) \Leftrightarrow AA^+ = A^+A$. For $A \in F[\lambda]^{nxn}$, a generalized inverse (or) {1} inverse is defined as a solution of the polynomial matrix equation (1.1) and a reflexive generalized inverse (or) {1,2} inverse is defined as a solution of the equations (1.1) and (1.2) and they belong to $F[\lambda]^{nxn}$. The purpose of this paper is to give a characterization of an EP_r- λ -matrix. Some results on EP_r-λ-marices having the same range space are obtained. As an application necessary and sufficient conditions are derived for $(AB)^+$ to be an EP_r-λ-matrix whenever A and B are EP_r-λ-matrices.

2. CHARACTERIZATION OF AN EP_r-λ-MATRIX

THEOREM 1. $A \in F_r^{n\times n}[\lambda]$ is EP_r over the field $F(\lambda)$ if and only if there exist an nxn unimodular λ-matrix P and a r x r regular λ-matrix E such that

$$PAP^* = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$$

PROOF. By the Smith's canonical form, $A = P\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q$ where P and Q are unimodular-λ-matrices of order n and D is a rxr regular diagonal λ-matrix. Any {1} inverse of A is given by $A^{(1)} = Q^{-1}\begin{bmatrix} D^{-1} & R_2 \\ R_3 & R_4 \end{bmatrix} P^{-1}$ where R_2, R_3, and R_4 are arbitrary conformable matrices over $F(\lambda)$. A is EP_r over the field $F(\lambda)$

$$\Rightarrow R(A) = R(A^*)$$

$$\Rightarrow A = AA^{*(1)}A^* \qquad \text{(By Theorem 17[3])}$$

$$\Rightarrow \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} QP^{*^{-1}} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} QP^{*^{-1}} \begin{bmatrix} D^{*^{-1}} & R_3^* \\ R_2^* & R_4^* \end{bmatrix} Q^{*^{-1}} Q^* \begin{bmatrix} D^* & 0 \\ 0 & 0 \end{bmatrix}$$

Partitioning conformably, let, $QP^{*^{-1}} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\begin{bmatrix} D^{*^{-1}} & R_3^* \\ R_2^* & R_4^* \end{bmatrix}\begin{bmatrix} D^* & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} DT_1 & DT_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} DT_1 + DT_2R_2^*D^* & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow T_2 = 0 \quad \text{(since D is regular)}.$$

Therefore $QP^{*^{-1}} = \begin{bmatrix} T_1 & 0 \\ T_3 & T_4 \end{bmatrix}$

Hence $A = P\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 & 0 \\ T_3 & T_4 \end{bmatrix} P^* = P\begin{bmatrix} DT_1 & 0 \\ 0 & 0 \end{bmatrix} P^* = P\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} P^*$

where $E = DT_1$ is a r x r regular λ-matrix.

Conversely, let $PAP^* = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ where E is a r x r regular λ-matrix.

Since E is regular, E is EP_r over $F(\lambda)$.

$$\Rightarrow R(E) = R(E^*)$$

$$\Rightarrow R(PAP^*) = R(PA^*P^*)$$

$$\Rightarrow R(A) = R(A^*)$$

$\Rightarrow$ A is EP_r over $F(\lambda)$. Hence the theorem.

If $A \in F_r^{n\times n}[\lambda]$ and is EP over the field $F(\lambda)$ then we can find nxn regular rational λ-matrices H and K such that $A^* = HA = AK$ [4]. In general the above H and K need not be unimodular λ-matrices. For example, consider $A = \begin{bmatrix} 1 & \lambda \\ 0 & \lambda^2 \end{bmatrix}$. A is

EP, being a regular λ-matrix. If $A^* = HA$ then $H = A^*A^{-1}$; If $A^* = AK$ then $K = A^{-1}A^*$. Here $H = \begin{bmatrix} 1 & -1/\lambda \\ \lambda & 0 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & -\lambda \\ 1/\lambda & 1 \end{bmatrix}$ are not λ-matrices.

The following theorem gives a necessary condition for H and K to be unimodular λ-matrices.

THEOREM 2. If A is an nxn EP_r-λ-matrix and A has a λ-matrix {1} inverse then there exist nxn unimodular λ-matrices H and K such that $A^* = HA = AK$.

PROOF. Let A be an nxn EP_r-λ-matrix. By Theorem 1, there exists an nxn unimodular λ-matrix P such that $PAP^* = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ where E is a rxr regular λ-matrix. Since A has a λ-matrix {1} inverse, E^{-1} is also a λ-matrix.

Now $$A = P^{-1} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} {P^{-1}}^*$$

Therefore $$A^* = P^{-1} \begin{bmatrix} E^* & 0 \\ 0 & 0 \end{bmatrix} {P^{-1}}^*$$

$$= P^{-1} \begin{bmatrix} E^*E^{-1} & 0 \\ 0 & I \end{bmatrix} PP^{-1} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} {P^{-1}}^*$$

$$= HA \text{ where } H = P^{-1} \begin{bmatrix} E^*E^{-1} & 0 \\ 0 & I \end{bmatrix} P$$ is an nxn unimodular λ-matrix. Similarly we can write $A^* = AK$ where

$K = P^* \begin{bmatrix} E^{-1}E^* & 0 \\ 0 & I \end{bmatrix} {P^{-1}}^*$ is an nxn unimodular λ-matrix.

Therefore $A^* = HA = AK$.

REMARK 1. The converse of Theorem 2 need not be true. For example, consider $A = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$. Since $A^* = A$, $H = K = I_2$. A is an EP_1-λ-matrix. However A has no λ-matrix {1} inverse.

3. MOORE-PENROSE INVERSE OF AN EP_r-λ-MATRIX

The following theorem gives a set of necesssry and sufficient conditions for the existence of the λ-matrix Moore-Penrose inverse of a given λ-matrix.

THEOREM 3. For $A \in F_r^{nxn}[\lambda]$, the following statements are equivalent.

i) A is EP_r, $rk(A) = rk(A^2)$ and A^*A has a λ-matrix {1} inverse.

ii) There exists an unimodular λ-matrix U with $A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$ where D is a rxr unimodular λ-matrix and U^*U is a diagonal block matrix.

iii) $A = GLG^*$ where L and G^*G are rxr unimodular λ-matrices and G is a λ-matrix.

iv) A^+ is a λ-matrix and EP_r.

v) There exists a symmetric idempotent λ-matrix E, ($E^2 = E = E^*$) such that $AE = EA$ and $R(A) = R(E)$.

PROOF. (i) $\Rightarrow$ (ii) Since A is an EP_r-λ-matrix over the field $F(\lambda)$ and $rk(A) = rk(A^2)$, A^+ exists, by Theorem 2.3 of [5]. By Theorem 4 in [6], A^*A has a λ-matrix {1} inverse implies that there exists an unimodular λ-matrix P with $PP^* = \begin{bmatrix} P_1 & 0 \\ 0 & P_4 \end{bmatrix}$ where P_1 is a symmetric rxr unimodular λ-matrix such that

$PA = \begin{bmatrix} W \\ 0 \end{bmatrix}$ where W is a rxn, λ-matrix of rank r. Hence by Theorem 2 in [6], AA^+ is a λ-matrix and $PAA^+P^* = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$. Since A is EP_r, $AA^+ = A^+A$ and $A = AA^+A = A(AA^+)$. Therefore $A = P^{-1}\begin{bmatrix} W \\ 0 \end{bmatrix} P^{-1}\begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} P^{*-1}$

$$= P^{-1}\begin{bmatrix} W \\ 0 \end{bmatrix}\begin{bmatrix} H & 0 \end{bmatrix} P^{*-1}$$ where

H consists of the first r columns of P^*, thus H is a nxr, λ-matrix of rank r.

Now $A = P^{-1}\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} P^{-1^*} = U\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$ where $U = P^{-1}$ and $D = WH$ is a rxr regular λ-matrix. Since A^*A has a λ-matrix {1} inverse and P is an unimodular λ-matrix, $PAA^*P^* = \begin{bmatrix} D^*P_1^{-1}D & 0 \\ 0 & 0 \end{bmatrix}$ has a λ-matrix {1} inverse. Therefore by Theorem 1 in [6], $D^*P_1^{-1}D$ is an unimodular λ-matrix which implies D is an unimodular λ-matrix. Hence $A = U\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}U^*$ where D is a rxr unimodular λ-matrix and U^*U is a diagonal block λ-matrix.

Thus (ii) holds.

(ii) ⇒ (iii)

Let us partition U as $U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$ where U_1 is a rxr λ-matrix. Then

$$A = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} U_1^* & U_3^* \\ U_2^* & U_4^* \end{bmatrix} = \begin{bmatrix} U_1 \\ U_3 \end{bmatrix} D \begin{bmatrix} U_1^* & U_3^* \end{bmatrix} = GLG^*$$

where $L = D$ and $G = \begin{bmatrix} U_1 \\ U_3 \end{bmatrix}$ are λ-matrices.

Since U^*U is a diagonal block λ-matrix, $G^*G = U_1^*U_1 + U_3^*U_3$ and L are rxr unimodular λ-matrices. Thus (iii) holds.

(iii) ⇒ (iv)

Since $A = GLG^*$, L and G^*G are unimodular λ-matrices. One can verify that $A^+ = G(G^*G)^{-1}L^{-1}(G^*G)^{-1}G^*$.

Now $AA^+ = GLG^*G(G^*G)^{-1}L^{-1}(G^*G)^{-1}G^* = G(G^*G)^{-1}G^* = A^+A$ implies that A^+ is EP_r. Since L and G^*G are unimodular, L^{-1} and $(G^*G)^{-1}$ are λ-matrices, and G is a λ-matrix. Therefore A^+ is a λ-matrix. Thus (iv) holds.

(iv) ⇒ (v)

Proof is analogous to that of (ii) ⇒ (iii) of Theorem 2.3 [5].

(v) ⇒ (i)

Since E is a symmetric idempotent λ-matrix with R(A) = R(E) and AE = EA, by Theorem 2.3 in [5] we have A is EP_r and $rk(A) = rk(A^2)$ ⇒ A^+ exists. Since $E^+ = E$ and $R(A) = R(E)$ ⇒ $AA^+ = EE^+ = E$. Now $AE = EA = (AA^+)A = A$. Let e_j and a_j denote the jth columns of E and A respectively. Then $AE = A$ ⇒ $Ae_j = a_j$, since e_j is a λ-matrix, the equation $Ax = a_j$ where a_j is a λ-matrix, has a λ-matrix solution. Hence by Theorem 1 in [6] it follows that A has a λ-matrix {1} inverse. Further $AA^+ = E$ is also a λ-matrix. Hence by Theorem 4 in [6] we see that A^*A has a λ-matrix {1} inverse. Thus (i) holds. Hence the theorem.

REMARK 2. The condition (i) in Theorem 3 cannot be weakened which can be seen by the following examples.

EXAMPLE 1. Consider the matrix $A = \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix}$. A is EP_1 and $rk(A) = rk(A^2) = 1$. $A^*A = \begin{bmatrix} 2\lambda^2 & 2\lambda^2 \\ 2\lambda^2 & 2\lambda^2 \end{bmatrix}$ has no λ-matrix {1} inverse (since the invariant polynomial of A^*A is λ^2 which is not the identity of F). For this A, $A^+ = \frac{1}{4\lambda}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not a λ-matrix. Thus the theorem fails.

EXAMPLE 2. Consider the matrix $A = \begin{bmatrix} \lambda & 2\lambda \\ 2\lambda & 4\lambda \end{bmatrix}$ over GF(5). A is EP_1. Since $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $rk(A) \neq rk(A^2)$, $A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has a λ-matrix {1} inverse (since any conformable λ-matrix is a λ-matrix {1} inverse). For this A, A^+ does not exist. Thus the theorem fails.

REMARKS 3. From Theorem 3, it is clear that if E is a symmetric idempotent λ-matrix, and A is a λ-matrix such that R(E) = R(A) then A is EP $\Leftrightarrow$ AE = EA $\Leftrightarrow$ A^+ is a λ-matrix and EP.

We can show that the set of all EP_r-λ-matrices with common range space as that of given symmetric idempotent λ-matrix forms a group, analogous to that of the Theorem 2.1 in [5].

COROLLARY 1. Let $E = E^* = E^2 \in F^{n\times n}_{[\lambda]}$. Then $H(E)=\{A \in F^{n\times n}_{[\lambda]}$: A is EP_r over $F(\lambda)$ and $R(A) = R(E)\}$ is s maximal subgroup of $F^{n\times n}_{[\lambda]}$ containing E as identity.

PROOF. This can be proved similar to that of Theorem 2.1 of [5] by applying Theorem 3.

4. APPLICATION

In general, if A and B are λ-matrices, having λ-matrix {1} inverses, it is not necesssary that AB has a λ-matrix {1} inverse.

EXAMPLE 3. Consider $A = \begin{bmatrix} 1 & \lambda \\ \lambda & \lambda^2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2\lambda & 0 \end{bmatrix}$. Here $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is one of the λ-matrix {1} inverse for both A and B. But $AB = \begin{bmatrix} 1+2\lambda^2 & 0 \\ \lambda+2\lambda^3 & 0 \end{bmatrix}$. Since the invariant polynomial of AB is $1+2\lambda^2 \neq 1$, AB has no λ-matrix {1} inverse.

The following theorem leads to the existence of λ-matrix {1} inverse of the product AB.

THEOREM 4. Let $A, B \in F^{n\times n}_{[\lambda]}$. If $A^2 = A$ and B has λ-matrix {1} inverse and $R(A) \subseteq R(B)$ then AB has a λ-matrix { 1 } inverse.

PROOF. Suppose ABx = b, where b is a λ-matrix, is a consistent system. Then $b \in R(AB) \subseteq R(A) \subseteq R(B)$ and therefore $Bz_0 = b$. Since B has a λ-matrix { 1 } inverse, by Theorem 1 in [6] we get z_0 is a λ-matrix. Since A is idempotent, so in particular A is a {1 }inverse of A and $b \in R(A)$, we have Ab=b. Now ABz_0 = Ab = b. Thus ABx = b has a λ-matrix solution. Hence by Theorem 1 in [6], AB has a λ-matrix {1} inverse. Hence the theorem.

The converse of Theorem 4 need not be true which can be seen by the following example.

EXAMPLE 4. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 \\ \lambda & \lambda \end{bmatrix}$; $AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Here $A^2 = A$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a λ-matrix {1} inverse for both AB and B. However

$R(A) \nsubseteq R(B)$. Hence the converse is not true.

Next we shall discuss the necessary and sufficient condition for the Moore-Penrose inverse of the product of EP_r-λ-matrices to be an EP_r-λ-matrix.

THEOREM 5. Let A and B be EP_r-λ-matrices. Then A^*A has a λ-matrix {1} inverse, $rk(A) = rk(A^2)$ and $R(A) = R(B)$ if and only if AB is EP_r and $(AB)^+ = B^+A^+$ is a λ-matrix.

PROOF. Since A and B are EP_r with $R(A) = R(B)$ and $rk(A) = rk(A^2)$, by a Theorem of Katz [1], AB is EP_r. Since A is a EP_r-λ-matrix, $rk(A) = rk(A^2)$ and A^*A has a λ-matrix {1} inverse, by Theorem 3, A^+ is a λ-matrix and there exists a symmetric idempotent λ-matrix E such that $R(A) = R(E)$. Hence $AA^+ = AA^+ = E$. Since A and B are EP_r and $R(A) = R(B)$, we have $AA^+ = BB^+ = E = A^+A = B^+B$. Therefore $BE = EB$ and $R(B) = R(E)$. Again from Theorem 3, for the EP_r-λ-matrix B, we see that B^+ is a λ-matrix. Since A and B are EP_r with $R(A) = R(B)$, we can verify that $(AB)^+ = B^+A^+$. Since B^+ and A^+ are λ-matrices, it follows that $(AB)^+$ is a λ-matrix.

Conversely, if $(AB)^+$ is a λ-matrix and AB is EP_r then $(AB)^+$ is an EP_r-λ-matrix. Therefore by Theorem 3, there exists a symmetric idempotent λ-matrix E such that $R(AB) = R(E)$ and $(AB)(AB)^+ = E = (AB)^+(AB)$. Since $rk(AB) = rk(A) = r$ and $R(AB) \subseteq R(A)$, we get $R(A) = R(E)$. Since A is EP_r, by Remark 3, it follows that A^+ is a EP_r-λ-matrix. Now by Theorem 3, A^*A has a λ-matrix {1} inverse and $rk(A) = rk(A^2)$. Since AB and B are EP_r, $R(E) = R(AB) = R((AB)^*) \subseteq R(B^*) = R(B)$ and $rk(AB) = rk(B)$ implies $R(B) = R(E)$. Therefore $R(A) = R(B)$. Hence the theorem.

REMARK 4. The condition that both A and B are EP_r-λ-matrices, is essential in Theorem 5, is illustrated as follows:

Let $A = \begin{bmatrix} 1 & \lambda \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2\lambda \\ 0 & 0 \end{bmatrix}$. A and B are not EP_1. $A^*A = \begin{bmatrix} 1 & \lambda \\ \lambda & \lambda^2 \end{bmatrix}$ has a λ-matrix {1} inverse and $R(A) = R(B)$. But AB is not EP_1. $(AB)^+ = \frac{1}{1+4\lambda^2}\begin{bmatrix} 1 & 0 \\ 2\lambda & 0 \end{bmatrix}$ is not a λ-matrix. Hence the claim.

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