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# MEASURES ON COALLOCATION AND NORMAL LATTICES

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## ABSTRACT

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be lattices of subsets of a nonempty set $X$. Suppose $\mathscr{L}_2$ coallocates $\mathscr{L}_1$ and $\mathscr{L}_1$ is a subset of $\mathscr{L}_2$. We show that any $\mathscr{L}_1$-regular finitely additive measure on the algebra generated by $\mathscr{L}_1$ can be uniquely extended to an $\mathscr{L}_2$-regular measure on the algebra generated by $\mathscr{L}_2$. The case when $\mathscr{L}_1$ is not necessary contained in $\mathscr{L}_2$, as well as the measure enlargement problem are considered. Furthermore, some discussions on normal lattices and separation of lattices are also given.

**KEY WORDS :** lattices, normal lattices, coallocation lattices, semi-separated lattices, regular finitely additive measures, $\sigma$-smooth measures, measure extension, measure enlargement.

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## 1. INTRODUCTION

Let $X$ be an arbitrary set and $\mathscr{L}_1$ and $\mathscr{L}_2$ are lattices of subsets of $X$. If $\mathscr{L}_1 \subset \mathscr{L}_2$, and if $\mathscr{L}_2$ coallocates $\mathscr{L}_1$, then any $\mathscr{L}_1$-regular finitely additive measure on the algebra generated by $\mathscr{L}_1$ can be uniquely extended to an $\mathscr{L}_2$-regular measure on the algebra generated by $\mathscr{L}_2$. This situation has been investigated by J. Camacho in [2]. We extend his results in several directions in this paper. We will consider the case where $\mathscr{L}_1$ is not necessary contained in $\mathscr{L}_2$ (see Theorem 3.1) and show that under suitable conditions any $\mu \in M_R(\mathscr{L}_1)$ (see below for definitions) gives rise to a $\nu \in M_R(\mathscr{L}_2)$. We will also

consider besides measure extension problems, measure enlargement problems (see e.g. Theorem 3.3) and will finally apply these results to the case of a single lattice $\mathscr{L}$, thereby extending results of M. Szeto [8] for measures on normal lattices.

We begin by giving some standard lattice and measure theoretic background in Section 2. Our notation and terminology is consistent with [1,4,6,7,9]. In Section 3, we consider the general coallocation theorem and a variety of consequences of it. Section 4 is devoted to a more detailed discussion of normal lattices and to separation of lattices. This work extends to some extent that of G. Eid [3].

## 2. BACKGROUND AND TERMINOLOGY

In this section, we summarize some lattice and measure theoretic notions and notations. This is all fairly standard and as previously mentioned is consistent with standard references.

**Definition 2.1**

Let $X$ be a nonempty set and $\mathscr{P}(X)$ is the power set of $X$. A **lattice** $\mathscr{L}$ is a collection of subsets of $X$, which is closed under finite unions and finite intersections, and $\emptyset, X \in \mathscr{L}$. Let

$$\mathscr{L}' \equiv \{ L' : L \in \mathscr{L} \}$$

where $L'$ denotes the **complement** of $L$. $\mathscr{L}'$ is a lattice if $\mathscr{L}$ is.

**Definition 2.2**

Let $\mathscr{L}$, $\mathscr{L}_1$ and $\mathscr{L}_2$ be any lattices of subsets of $X$.

(1) $\mathscr{L}$ is $\delta$ if it is closed under countable intersections.

(2) $\mathscr{L}$ is a **complement generated (c.g.) lattice** if

$\forall L \in \mathscr{L}$, $\exists L_1, L_2, \ldots \in \mathscr{L}$ such that $L = \bigcap_{n=1}^{\infty} L_n'$.

(3) $\mathscr{L}$ is a **normal lattice** if

$\forall L_1, L_2 \in \mathscr{L}$, $L_1 \cap L_2 = \emptyset \Rightarrow$

$\exists \tilde{L}_1, \tilde{L}_2 \in \mathscr{L}$ s.t. $L_1 \subset \tilde{L}_1'$, $L_2 \subset \tilde{L}_2'$, $\tilde{L}_1' \cap \tilde{L}_2' = \emptyset$.

(4) $\mathscr{L}$ is a **countably paracompact (c.p.) lattice** if

$\forall L_1, L_2, \ldots \in \mathscr{L}$, $L_1 \supset L_2 \supset \ldots$, $\lim_{n\to\infty} L_n = \emptyset$ $(L_n \downarrow \emptyset) \Rightarrow$

$\exists \tilde{L}_1, \tilde{L}_1, \ldots \in \mathscr{L}$ s.t. $\forall n$, $L_n \subset \tilde{L}_n'$ and $\tilde{L}_n' \downarrow \emptyset$.

(5) $\mathscr{L}_2$ is **$\mathscr{L}_1$-countably-paracompact ($\mathscr{L}_1$-c.p.)** if

$\forall B_1, B_2, \ldots \in \mathscr{L}_2$, $B_1 \supset B_2 \supset \ldots$, $B_n \downarrow \emptyset \quad \Rightarrow$

$\exists A_1, A_2, \ldots \in \mathscr{L}_1$ s.t. $\forall n$, $B_n \subset A_n'$ and $A_n' \downarrow \emptyset$.

(6) $\mathscr{L}_1$ **semi-separates** $\mathscr{L}_2$ if

$\forall A \in \mathscr{L}_1$, $B \in \mathscr{L}_2$, $A \cap B = \emptyset \Rightarrow \exists \tilde{L}_1 \in \mathscr{L}_1$, s.t. $B \subset \tilde{L}_1$ and $A \cap \tilde{L}_1 = \emptyset$.

(7) $\mathscr{L}_1$ **separates** $\mathscr{L}_2$ if

$\forall \tilde{L}_2, \hat{L}_2 \in \mathscr{L}_2$, $\tilde{L}_2 \cap \hat{L}_2 = \emptyset \Rightarrow$

$\exists \tilde{L}_1, \hat{L}_1 \in \mathscr{L}_1$, s.t. $\tilde{L}_2 \subset \tilde{L}_1$, $\hat{L}_2 \subset \hat{L}_1$, and $\tilde{L}_1 \cap \hat{L}_1 = \emptyset$.

(8) $\mathscr{L}_1$ **coseparates** $\mathscr{L}_2$ if

$\forall \tilde{L}_2, \hat{L}_2 \in \mathscr{L}_2$, $\tilde{L}_2 \cap \hat{L}_2 = \emptyset \Rightarrow$

$\exists \tilde{L}_1, \hat{L}_1 \in \mathscr{L}_1$, s.t. $\tilde{L}_2 \subset \tilde{L}_1'$, $\hat{L}_2 \subset \hat{L}_1'$, and $\tilde{L}_1' \cap \hat{L}_1' = \emptyset$.

(9) $\mathscr{L}_2$ **coallocates** $\mathscr{L}_1$ if

$\forall L_1 \in \mathscr{L}_1$ s.t. $L_1 \subset \tilde{L}_2' \cup \hat{L}_2'$, where $\tilde{L}_2, \hat{L}_2 \in \mathscr{L}_2 \Rightarrow$

$\exists \tilde{L}_1, \hat{L}_1 \in \mathscr{L}_1$ s.t. $\tilde{L}_1 \subset \tilde{L}_2'$, $\hat{L}_1 \subset \hat{L}_2'$, $L_1 = \tilde{L}_1 \cup \hat{L}_1$.

## Definition 2.3

A **finitely additive (f.a.) measure** $\mu$ is a finite nonnegative function defined on the algebra $A(\mathscr{L})$ generated by $\mathscr{L}$, such that (1) $\forall A \in A(\mathscr{L})$, $\mu(A) \geq 0$, (2) $\mu(\emptyset) = 0$, and (3) [*finite additivity*] $\forall A, B \in A(\mathscr{L})$, $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$.

A **0-1 measure** $\mu$ is a two-valued finitely additive measure taking value either 0 or 1.

Usually, we simply refer $\mu$ to as a *measure on a lattice* $\mathscr{L}$ to mean that $\mu$ is a finitely additive measure defined on the *algebra* $A(\mathscr{L})$.

A f.a. measure $\mu$ defined on the algebra $A(\mathscr{L})$ is

(1) $\mathscr{L}$-**regular** iff $\forall A \in A(\mathscr{L})$, $\mu(A) = \sup \{ \mu(L) : L \subset A, L \in \mathscr{L} \}$.

Or, equivalently, $\mu(A) = \inf \{ \mu(\hat{L}') : \hat{L}' \supset A, \hat{L} \in \mathscr{L} \}$.

(2) $\sigma$-**smooth on** $A(\mathscr{L})$, iff

$\forall A_1, A_2, \ldots \in A(\mathscr{L})$, $A_1 \supset A_2 \supset \ldots \downarrow \emptyset$, $(A_n \downarrow \emptyset) \Rightarrow \mu(A_n) \to 0$ as $n \to \infty$

(3) $\sigma$-**smooth on** $\mathscr{L}$, iff

$\forall L_1, L_2, \ldots \in \mathscr{L}$, $L_1 \supset L_2 \supset \ldots \downarrow \emptyset$, $(L_n \downarrow \emptyset) \Rightarrow \mu(L_n) \to 0$ as $n \to \infty$

The following notations for the collections of measures on $A(\mathscr{L})$ will be used throughout :

$M(\mathscr{L}) \equiv \{ \mu : \mu$ f.a. measure on $A(\mathscr{L}) \}$

$M_R(\mathscr{L}) \equiv \{ \mu \in M(\mathscr{L}) : \mu$ $\mathscr{L}$-regular $\}$

$M^\sigma(\mathscr{L}) \equiv \{ \mu \in M(\mathscr{L}) : \mu$ $\sigma$-smooth on $A(\mathscr{L}) \}$

$M_\sigma(\mathscr{L}) \equiv \{ \mu \in M(\mathscr{L}) : \mu$ $\sigma$-smooth on $\mathscr{L} \}$

$M_R^\sigma(\mathscr{L}) \equiv \{ \mu \in M(\mathscr{L}) : \mu$ $\sigma$-smooth on $A(\mathscr{L})$ and $\mathscr{L}$-regular $\}$

Similarly, we also define $I(\mathscr{L})$, $I_R(\mathscr{L})$, $I^\sigma(\mathscr{L})$, $I_\sigma(\mathscr{L})$, and $I_R^\sigma(\mathscr{L})$ for non-trivial 0-1 measures.

If $\mu$ is $\mathscr{L}$-regular, then $\sigma$-smoothness on $\mathscr{L}$ implies $\sigma$-smoothness on $A(\mathscr{L})$. Thus, $M_R^\sigma(\mathscr{L}) = M_R(\mathscr{L}) \cap M_\sigma(\mathscr{L}) = M_R(\mathscr{L}) \cap M^\sigma(\mathscr{L})$.
Since $A(\mathscr{L}') = A(\mathscr{L})$, we have $M(\mathscr{L}') = M(\mathscr{L})$ and $I(\mathscr{L}') = I(\mathscr{L})$.
Furthermore, $\mu$ is *$\sigma$-smooth on* $A(\mathscr{L})$ ($\mu \in M^\sigma(\mathscr{L})$) iff $\mu$ is *countably additive*.

Let $\mu_1, \mu_2 \in M(\mathscr{L})$. Define

(1) $\mu_1 \le \mu_2$ if $\forall A \in A(\mathscr{L})$, $\mu_1(A) \le \mu_2(A)$

(2) $\mu_1 \le \mu_2$ on $\mathscr{L}$, if $\forall L \in \mathscr{L}$, $\mu_1(L) \le \mu_2(L)$

(3) $\mu_1 \le \mu_2$ on $\mathscr{L}'$, if $\forall L \in \mathscr{L}$, $\mu_1(L') \le \mu_2(L')$

## Definition 2.4

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$ are lattices of subsets of $X$ such that $\mu_1 \in M(\mathscr{L}_1)$ and $\mu_2 \in M(\mathscr{L}_2)$. Denote $\mu_2|_{\mathscr{L}_1}$ (or simply $\mu_2|$) to mean the **restriction** of $\mu_2$ to $A(\mathscr{L}_1)$.

If $\mu_1 = \mu_2|$ on $A(\mathscr{L}_1)$, then $\mu_2$ is called a **measure extension** of $\mu_1$ from $A(\mathscr{L}_1)$ to $A(\mathscr{L}_2)$ (or, less precisely, from $\mathscr{L}_1$ to $\mathscr{L}_2$); and a **regular measure extension**, if $\mu_2 \in M_R(\mathscr{L}_2)$.

If $\mu_1 \le \mu_2|$ on $\mathscr{L}_1$ and $\mu_1(X) = \mu_2(X)$, then $\mu_2$ is called a **measure enlargement** of $\mu_1$ from $A(\mathscr{L}_1)$ to $A(\mathscr{L}_2)$ (or, less precisely, from $\mathscr{L}_1$ to $\mathscr{L}_2$); and a **regular measure enlargement**, if $\mu_2 \in M_R(\mathscr{L}_2)$.

## Definition 2.5

A real-valued function $\mu_* : \mathscr{P}(X) \to [0,\infty)$, is called a **finitely superadditive inner measure**, if

(1) $\mu_*(\emptyset) = 0$

(2) [*nondecreasing*] $\forall A \subset B \subset X \Rightarrow \mu_*(A) \le \mu_*(B)$, that is, $\mu_* \uparrow$

(3) [*finite superadditivity*] $\forall A,B \subset X$, $A \cap B = \emptyset \Rightarrow \mu_*(A \cup B) \ge \mu_*(A) + \mu_*(B)$

A real-valued function $\mu^* : \mathscr{P}(X) \to [0,\infty)$, is called a **finitely subadditive outer measure**, if it satisfies (1), (2) and

(3') [*finite subadditivity*] $\forall A,B \subset X$, $A \cap B = \emptyset \Rightarrow \mu^*(A \cup B) \le \mu^*(A) + \mu^*(B)$

Let $\mu^*$ be a finitely subadditive outer measure on $(X, \mathscr{L})$. A set $E \subset X$ is said to be **$\mu^*$-measurable**, if

$$\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E'), \quad \forall T \subset X .$$

We have the following theorem characterizing a normal lattice as a special case of the *coallocation property* :

**THEOREM 2.1**

$\mathscr{L}$ is *normal* $\Leftrightarrow$ $\forall L\in\mathscr{L}$ s.t. $L\subset L_1'\cup L_2'$, where $L_1,L_2\in\mathscr{L}\Rightarrow$
$\exists\,\tilde{L}_1,\tilde{L}_2\in\mathscr{L}$ s.t. $L_1\subset\tilde{L}_1'$, $L_2\subset\tilde{L}_2'$, $L=\tilde{L}_1\cup\tilde{L}_2$.

*Proof*:

"$\Leftarrow$"

Suppose $L_1,L_2\in\mathscr{L}$ and $L_1\cap L_2=\emptyset$. $\therefore X=L_1'\cup L_2'$. Then by assumption, $\exists\,\tilde{L}_1,\tilde{L}_2\in\mathscr{L}$ such that $\tilde{L}_1\subset L_1'$, $\tilde{L}_2\subset L_2'$ and $X=\tilde{L}_1\cup\tilde{L}_2$ or $\tilde{L}_1'\cap\tilde{L}_2'=\emptyset$.
Thus, when $L_1\cap L_2=\emptyset$, $L_1\subset\tilde{L}_1'$, $L_2\subset\tilde{L}_2'$, we have $\tilde{L}_1'\cap\tilde{L}_2'=\emptyset$. $\therefore\mathscr{L}$ is *normal*.

"$\Rightarrow$"

Let $L\in\mathscr{L}$ and $L\subset L_1'\cup L_2'$ where $L_1,L_2\in\mathscr{L}$. Consider $L-L_1'$ and $L-L_2'$, $(L-L_1')\cap(L-L_2') = (L\cap L_1)\cap(L\cap L_2) = L\cap(L_1'\cup L_2')' = \emptyset$. $\therefore L-L_1'$ and $L-L_2'$ are disjoint. By normality, $\exists\,\hat{L}_1,\hat{L}_2\in\mathscr{L}$, such that $(L-L_1')\subset\hat{L}_1'$, $(L-L_2')\subset\hat{L}_2'$ and $\hat{L}_1'\cap\hat{L}_2'=\emptyset$, or $\hat{L}_1\cup\hat{L}_2=X$. Define

$$\tilde{L}_1\equiv(L-\hat{L}_1') = L\cap\hat{L}_1\in\mathscr{L} \quad\text{and}\quad \tilde{L}_2\equiv(L-\hat{L}_2') = L\cap\hat{L}_2\in\mathscr{L}$$

Then $\tilde{L}_1\cup\tilde{L}_2 = (L\cap\hat{L}_1)\cup(L\cap\hat{L}_2) = L\cap(\hat{L}_1\cup\hat{L}_2) = L\cap X = L$. Now,

$$L-L_1'\subset\hat{L}_1' \Rightarrow \hat{L}_1\subset L'\cup L_1' \quad\text{and}\quad L-L_2'\subset\hat{L}_2' \Rightarrow \hat{L}_2\subset L'\cup L_2'$$

$\therefore$

$$\tilde{L}_1 = L\cap\hat{L}_1\subset L\cap(L'\cup L_1') = (L\cap L')\cup(L\cap L_1') = (L\cap L_1')\subset L_1'$$

and

$$\tilde{L}_2 = L\cap\hat{L}_2\subset L\cap(L'\cup L_2') = (L\cap L')\cup(L\cap L_2') = (L\cap L_2')\subset L_2'$$

Thus, $\forall L\in\mathscr{L}$, $L\subset L_1'\cup L_2'$, $\exists\,\tilde{L}_1,\tilde{L}_2\in\mathscr{L}$, s.t. $\tilde{L}_1\subset L_1'$, $\tilde{L}_2\subset L_2'$ and $L=\tilde{L}_1\cup\tilde{L}_2$. ∎

The following results are obvious :

(1) $\mathscr{L}$ *is normal* $\Leftrightarrow$ $\mathscr{L}$ *coallocates itself* $\Leftrightarrow$ $\mathscr{L}$ *coseparates itself*.

(2) $\mathscr{L}_1$ *separates* $\mathscr{L}_2$ $\Rightarrow$ $\mathscr{L}_1$ *semi-separates* $\mathscr{L}_2$.

Furthermore, we have the following measure theoretic characterization of a normal lattice :

**THEOREM 2.2**

$\mathscr{L}$ is *normal* iff
$\forall\,\mu\in I(\mathscr{L})$, s.t. on $\mathscr{L}$, $\mu\le\nu_1\in I_R(\mathscr{L})$, $\mu\le\nu_2\in I_R(\mathscr{L})$ $\Rightarrow$ $\nu_1=\nu_2$.

**THEOREM 2.3**

Suppose $\mathscr{L}_1\subset\mathscr{L}_2$. Then $\mathscr{L}_1$ *coseparates* $\mathscr{L}_2$ $\Rightarrow$ $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$.

*Proof*:

Suppose $L_1\subset\tilde{L}_2'\cup\hat{L}_2'$, where $L_1\in\mathscr{L}_1$, $\tilde{L}_2,\hat{L}_2\in\mathscr{L}$. Then, $L_1\cap\tilde{L}_2, L_1\cap\hat{L}_2\in\mathscr{L}_2\supset\mathscr{L}_1$.

Now $(L_1\cap\tilde{L}_2)\cap(L_1\cap\hat{L}_2) = L_1\cap(\tilde{L}_2\cap\hat{L}_2) = L_1\cap(\tilde{L}_2'\cup\hat{L}_2')' = \emptyset$.

$\mathscr{L}_1$ coseparates $\mathscr{L}_2\Rightarrow\exists\,\tilde{L}_1,\hat{L}_1\in\mathscr{L}_1$ s.t. $\tilde{L}_1'\cap\hat{L}_1'=\emptyset$, and

$L_1 \cap \tilde{L}_2 \subset \tilde{L}_1'$ and $L_1 \cap \hat{L}_2 \subset \hat{L}_1'$.

Define $\tilde{L}_1^\circ \equiv L_1 \cap \tilde{L}_1$ and $\hat{L}_1^\circ \equiv L_1 \cap \hat{L}_1$, hence $\tilde{L}_1^\circ, \hat{L}_1^\circ \in \mathscr{L}_1$. And

$$\tilde{L}_1^\circ \cup \hat{L}_1^\circ = (L_1 \cap \tilde{L}_1) \cup (L_1 \cap \hat{L}_1) = L_1 \cap (\tilde{L}_1 \cup \hat{L}_1) = L_1 \cap (\tilde{L}_1' \cap \hat{L}_1')' = L_1.$$

$\therefore L_1 = \tilde{L}_1^\circ \cup \hat{L}_1^\circ$. Now

$\tilde{L}_1^\circ = L_1 \cap \tilde{L}_1 \subset L_1 \cap (L_1 \cap \tilde{L}_2)' = L_1 \cap (L_1' \cup \tilde{L}_2') = (L_1 \cap L_1') \cup (L_1 \cap \tilde{L}_2')$

$= L_1 \cap \tilde{L}_2' \subset \tilde{L}_2'$.

Thus, $\tilde{L}_1^\circ \subset \tilde{L}_2'$. Similarly, $\hat{L}_1^\circ \subset \hat{L}_2'$. Hence $\mathscr{L}_2$ coallocates $\mathscr{L}_1$. ∎

**THEOREM 2.4**

*$\mathscr{L}$ countably paracompact* $\Rightarrow$ $M_\sigma(\mathscr{L}') \subset M_\sigma(\mathscr{L})$

*Proof*:

Suppose $\forall n$, $L_n \in \mathscr{L}$, $L_n \downarrow \emptyset$

$\mathscr{L}$ c.p. $\Rightarrow$ $\exists \tilde{L}_n \in \mathscr{L}$, s.t. $L_n \subset \tilde{L}_n'$, $\tilde{L}_n' \downarrow \emptyset$

$\mu \in M_\sigma(\mathscr{L}')$ $\Leftrightarrow$ $\mu(\tilde{L}_n') \to 0$ as $\tilde{L}_n' \downarrow \emptyset$

$\therefore$ $\mu(L_n) \le \mu(\tilde{L}_n') \to 0$, $\Rightarrow$ $\mu(L_n) \to 0$

Now $L_n \downarrow \emptyset$, $\mu(L_n) \to 0$ $\therefore$ $\mu \in M_\sigma(\mathscr{L})$. Hence, $M_\sigma(\mathscr{L}') \subset M_\sigma(\mathscr{L})$. ∎

## 3. MEASURES ON COALLOCATION LATTICES

In this section we extend some of the work of [8] and [2] on the unique extendability of a measure $\mu \in M_R(\mathscr{L}_1)$ to a measure $\nu \in M_R(\mathscr{L}_2)$ where $\mathscr{L}_1$ and $\mathscr{L}_2$ are lattices of subsets of $X$. We note that it is not always necessary to assume that $\mathscr{L}_1 \subset \mathscr{L}_2$ nor that $X$ belongs to the lattices in order for the main results of the **coallocation theorem** to hold (see Theorem 3.1). We first define two functions which form an inner-outer measure pair.

**Definition 3.1**

Suppose $\mathscr{L}_1$ and $\mathscr{L}_2$ are lattices of subsets of $X$ and $\mu \in M(\mathscr{L}_1)$. For all $E \subset X$, define

$$\mu_\cdot(E) \equiv \sup \{ \mu(L_1) : E \supset L_1,\ L_1 \in \mathscr{L}_1 \}$$

and

$$\mu^\wedge(E) \equiv \inf \{ \mu_\cdot(L_2') : E \subset L_2',\ L_2 \in \mathscr{L}_2 \}$$

We have the following :

**THEOREM 3.1** [*Coallocation theorem*]

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be lattices of subsets of $X \neq \emptyset$. Suppose $\mu \in M(\mathscr{L}_1)$. We have

(1) $\mu_*$ is a **finitely superadditive inner measure**

(2) $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$ $\Rightarrow$ $\mu_*$ is finitely additive on $\mathscr{L}_2'$

(3) $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$ $\Rightarrow$ $\mu^\wedge$ is a **finitely subadditive outer measure**

(4) $\mu^\wedge = \mu_*$ *on* $\mathscr{L}_2'$

In particular, if $X \in \mathscr{L}_1$ *and* $\emptyset \in \mathscr{L}_2$, then $\mu^\wedge(X) = \mu_*(X) = \mu(X)$

(5) [a] $\mu \le \mu^\wedge$ *on* $\mathscr{L}_1$

[b] $\mathscr{L}_1 \subset \mathscr{L}_2$ *and* $\mu \in \mathsf{M}_R(\mathscr{L}_1)$ $\Rightarrow$ $\mu_* = \mu$ *on* $\mathscr{L}_1'$ ; $\mu^\wedge = \mu$ *on* $\mathscr{L}_1$

(6) Suppose $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$. $E \subset X$ is **$\mu^\wedge$-measurable**

$\Leftrightarrow$ $\forall L_2 \in \mathscr{L}_2$, $\mu^\wedge(L_2') \ge \mu^\wedge(L_2' \cap E) + \mu^\wedge(L_2' \cap E')$

(7) Suppose $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$.

If either [a] $\mathscr{L}_1 \subset \mathscr{L}_2$

or [b] $\mathscr{L}_2$ *semi-separates* $\mathscr{L}_1$

then

[1°] every element of $\mathscr{L}_2'$ is $\mu^\wedge$*-measurable*

[2°] $\mu^\wedge|_{\mathscr{L}_2}$ is a *finitely additive measure on* $\mathsf{A}(\mathscr{L}_2)$

[3°] $\mu^\wedge$ is $\mathscr{L}_1$*-regular on* $\mathscr{L}_2'$

[4°] $\mu^\wedge \in \mathsf{M}_R(\mathscr{L}_2)$.

*Proof:*

(1) The proof is standard and is therefore omitted.

(2) Let $A_2, B_2 \in \mathscr{L}_2$, and $L_1 \in \mathscr{L}$ s.t. $L_1 \subset A_2' \cup B_2' \in \mathscr{L}_2'$

$\mathscr{L}_2$ coallocates $\mathscr{L}_1$ $\Rightarrow$

$\exists\ A_1, B_1 \in \mathscr{L}_1$ s.t. $A_1 \subset A_2'$, $B_1 \subset B_2'$, and $L_1 = A_1 \cup B_1$

$$\begin{aligned} \therefore\ \mu(L_1) &= \mu(A_1 \cup B_1) \\ &\le \mu(A_1) + \mu(B_1) \\ &\le \sup\{\ \mu(A_1) : A_2' \supset A_1\ \} + \sup\{\ \mu(B_1) : B_2' \supset B_1\ \} \\ &\equiv \mu_*(A_2') + \mu_*(B_2') \end{aligned}$$

Taking sup on the left hand side,

$$\sup\{\ \mu(L_1) : L_1 \subset A_2' \cup B_2'\ \} \le \mu_*(A_2') + \mu_*(B_2')$$

$\therefore\ \mu_*(A_2' \cup B_2') \le \mu_*(A_2') + \mu_*(B_2')$ $\Rightarrow$

$\mu_*$ is finitely subadditive on $\mathscr{L}_2'$. Together with (1), $\mu_*$ is finitely additive on $\mathscr{L}_2'$.

(3) The proof is also standard and is omitted.

(4) Let $L_2 \in \mathscr{L}_2$.

$\mu^\wedge(L_2') = \inf\ \{\ \mu_*(\tilde{L}_2') : L_2' \subset \tilde{L}_2',\ \tilde{L}_2 \in \mathscr{L}_2\ \}$

$\Rightarrow$ $\mu^\wedge(L_2') \le \mu_*(L_2')$ .....[i]

Now if $A_2' \in \mathscr{L}_2'$ and $L_2' \subset A_2'$, then by monotonicity of $\mu_*$,

$\mu_*(L_2') \leq \mu_*(A_2')$

$\Rightarrow \mu_*(L_2') \leq \inf\{\mu_*(A_2') : L_2' \subset A_2',\ A_2 \in \mathscr{L}_2\} = \mu^\wedge(L_2')$ .....[ii]

[i] and [ii] $\Rightarrow$ $\mu^\wedge = \mu_*$ on $\mathscr{L}_2'$.

If $X \in \mathscr{L}_1$, take $L_1 = X$, $\mu_*(X) = \sup\{\mu(L_1) : X \supset L_1 \in \mathscr{L}_1\} = \mu(X)$.

If $\emptyset \in \mathscr{L}_2$, $X = \emptyset' \in \mathscr{L}_2'$ and $\mu^\wedge = \mu_*$ on $\mathscr{L}_2'$ $\Rightarrow$ $\mu^\wedge(X) = \mu_*(X)$

Consequently, $\mu^\wedge(X) = \mu_*(X) = \mu(X)$.

(5)[a] Let $L_1 \in \mathscr{L}_1$ and $A_2 \in \mathscr{L}_2$, s.t. $L_1 \subset A_2'$

$\mu_*(A_2') = \sup\{\mu(L_1) : A_2' \supset L_1 \in \mathscr{L}_1\} \geq \mu(L_1)$

Taking inf, $\inf\{\mu_*(A_2') : L_1 \subset A_2',\ A_2 \in \mathscr{L}_2\} \geq \mu(L_1)$

i.e. $\mu^\wedge(L_1) \geq \mu(L_1)$. Or, $\mu \leq \mu^\wedge$ on $\mathscr{L}_1$.

(5)[b] Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$, then $\mu^\wedge = \mu_*$ on $\mathscr{L}_2'$ $\Rightarrow$ $\mu^\wedge = \mu_*$ on $\mathscr{L}_1'$

$\therefore$ if $\tilde{L}_1 \in \mathscr{L}_1$, then $\mu^\wedge(\tilde{L}_1') = \mu_*(\tilde{L}_1')$

Suppose $L_1 \subset A_2' \subset A_1'$, where $A_1 \in \mathscr{L}_1$, $A_2 \in \mathscr{L}_2$, $A_1 \subset A_2$

$\mu_*(A_1') = \sup\{\mu(\tilde{L}_1) : A_1' \subset \tilde{L}_1 \in \mathscr{L}_1\}$

But $\mu \in M_R(\mathscr{L}_1)$ $\Rightarrow \mu(A_1') = \sup\{\mu(\tilde{L}_1) : \tilde{L}_1 \in \mathscr{L}_1\}$ whenever $A_1' \subset \tilde{L}_1$

$\therefore \mu_*(A_1') = \mu(A_1')$ $\forall A_1' \in \mathscr{L}_1'$, hence $\mu_* = \mu$ on $\mathscr{L}_1'$.

$$\begin{aligned}
\text{Now, } \mu^\wedge(L_1) &= \inf\{\mu_*(A_2') : L_1 \subset A_2',\ A_2 \in \mathscr{L}_2\} \\
&\leq \inf\{\mu_*(A_1') : L_1 \subset A_1',\ A_1 \in \mathscr{L}_1\} && (\because A_2' \subset A_1') \\
&= \inf\{\mu(A_1') : L_1 \subset A_1',\ A_1 \in \mathscr{L}_1\} && (\because \mu_* = \mu \text{ on } \mathscr{L}_1') \\
&= \mu(L_1) && (\because \mu \in M_R(\mathscr{L}_1))
\end{aligned}$$

$\therefore \mu^\wedge \leq \mu$ on $\mathscr{L}_1$, and (5)[a] $\Rightarrow$ $\mu \leq \mu^\wedge$ on $\mathscr{L}_1$, $\therefore$ $\mu^\wedge = \mu$ on $\mathscr{L}_1$.

(6) "$\Leftarrow$" Suppose $\forall A_2 \in \mathscr{L}_2$, we have $\forall E \subset X$,

$\mu^\wedge(A_2') \geq \mu^\wedge(A_2' \cap E) + \mu^\wedge(A_2' \cap E')$

Suppose $T \subset X$, s.t. $T \subset A_2'$, $A_2 \in \mathscr{L}_2$

$\mu^\wedge(T) = \inf\{\mu_*(A_2') : T \subset A_2',\ A_2 \in \mathscr{L}_2\}$

Now (4) $\Rightarrow$

$$\begin{aligned}
\mu_*(A_2') &= \mu^\wedge(A_2') \\
&\geq \mu^\wedge(A_2' \cap E) + \mu^\wedge(A_2' \cap E') && \text{(by assumption)} \\
&\geq \mu^\wedge(T \cap E) + \mu^\wedge(T \cap E') && (\because T \subset A_2',\ \mu^\wedge \uparrow)
\end{aligned}$$

Taking inf,

$\mu^\wedge(T) \geq \mu^\wedge(T \cap E) + \mu^\wedge(T \cap E')$ ..... [iii]

$\mu^\wedge$ is finitely subadditive $\Rightarrow$

$\mu^\wedge(T) = \mu^\wedge(T \cap (E \cup E')) \leq \mu^\wedge(T \cap E) + \mu^\wedge(T \cap E')$ ..... [iv]

[iii] and [iv] $\Rightarrow$ $\mu^\wedge(T) = \mu^\wedge(T\cap E) + \mu^\wedge(T\cap E')$ $\forall T \subset X$,
which is the definition of $E$ to be $\mu^\wedge$-measurable.

(6) "$\Rightarrow$" By the definition of $E$ to be $\mu^\wedge$-measurable, we have

$$\mu^\wedge(T) = \mu^\wedge(T\cap E) + \mu^\wedge(T\cap E') \quad \forall T \subset X,$$

But $\mu^\wedge$ is finitely subadditive, the above is equivalent to

$$\mu^\wedge(T) \geq \mu^\wedge(T\cap E) + \mu^\wedge(T\cap E') \quad \forall T \subset X,$$

In particular, take $T = L_2' \in \mathscr{L}_2'$, we have

$$\mu^\wedge(L_2') \geq \mu^\wedge(L_2'\cap E) + \mu^\wedge(L_2'\cap E') \quad \forall L_2 \in \mathscr{L}_2 .$$

(7) Suppose $\mathscr{L}_2$ coallocates $\mathscr{L}_1$.
Let $L_2' \in \mathscr{L}_2'$. To prove that $L_2'$ is $\mu^\wedge$-measurable, we have to show, by (6), that

$$\mu^\wedge(A_2') \geq \mu^\wedge(A_2'\cap L_2') + \mu^\wedge(A_2'\cap L_2) \quad \forall A_2 \in \mathscr{L}_2 .$$

$\forall A_2 \in \mathscr{L}_2$, let $P, Q \in \mathscr{L}_1$ ($\therefore P\cup Q \in \mathscr{L}_1$) s.t.

$$P \subset A_2'\cap L_2' \quad \text{and} \quad Q \subset A_2'\cap P'$$

Thus, $P \subset A_2'$ and $Q \subset A_2'$

Now, $P \cup Q \subset (A_2'\cap L_2') \cup (A_2'\cap P') \subset A_2'$

and $P \cap Q \subset P \cap (A_2'\cap P') = \emptyset$

$$\begin{aligned}
\mu^\wedge(A_2') &= \mu_*(A_2') \qquad (\mu^\wedge = \mu_* \text{ on } \mathscr{L}_2') \\
&\geq \sup\{\, \mu(P\cup Q) : A_2' \supset P\cup Q \in \mathscr{L}_1 \,\} \\
&\geq \mu(P\cup Q) \\
&= \mu(P) + \mu(Q) \qquad (P\cap Q = \emptyset)
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\mu^\wedge(A_2') &\geq \mu(P) + \sup\{\, \mu(Q) : A_2'\cap P' \supset Q \in \mathscr{L}_1 \,\} \\
&= \mu(P) + \mu_*(A_2'\cap P')
\end{aligned}$$

$\Rightarrow$

$$\mu^\wedge(A_2') \geq \mu(P) + \mu_*(A_2'\cap P') \qquad \text{.....[v]}$$

(7)[a] : Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$.

If $\mathscr{L}_1 \subset \mathscr{L}_2$, then $P \in \mathscr{L}_1 \Rightarrow P \in \mathscr{L}_2$ $\therefore A_2 \cup P \in \mathscr{L}_2$

$\Rightarrow$

$A_2'\cap P' = (A_2 \cup P)' \in \mathscr{L}_2'$, and $\mu^\wedge = \mu_*$ on $\mathscr{L}_2'$, $\therefore$ [v] $\Rightarrow$

$$\begin{aligned}
\mu^\wedge(A_2') &\geq \mu(P) + \mu^\wedge(A_2'\cap P') \\
&\geq \mu(P) + \mu^\wedge(A_2'\cap L_2) \qquad (P \subset L_2')
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\mu^\wedge(A_2') &\geq \sup\{\, \mu(P) : A_2'\cap L_2' \supset P \in \mathscr{L}_1 \,\} + \mu^\wedge(A_2'\cap L_2) \\
&\equiv \mu_*(A_2'\cap L_2') + \mu^\wedge(A_2'\cap L_2) \\
&= \mu^\wedge(A_2'\cap L_2') + \mu^\wedge(A_2'\cap L_2) \qquad \forall A_2 \in \mathscr{L}_2 \ (\mu^\wedge = \mu_* \text{ on } \mathscr{L}_2')
\end{aligned}$$

We conclude, from (6), that every element of $\mathscr{L}_2'$ is $\mu^\wedge$-measurable.
$\therefore A(\mathscr{L}_2) = A(\mathscr{L}_2') \subset \{\, \mu^\wedge\text{-measurable sets} \,\}$. By a standard Carathéodory

argument, $\mu^\wedge|_{\mathscr{L}_2}$ is a *finitely additive measure on* $A(\mathscr{L}_2)$.

Suppose $L_2 \in \mathscr{L}_2$,

$$\begin{aligned}\mu^\wedge(L_2') &= \mu_*(L_2') && (\text{ by (4) })\\ &= \sup\{\mu(L_1) : L_1 \subset L_2',\ L_1 \in \mathscr{L}_1\}\\ &\le \sup\{\mu^\wedge(L_1) : L_1 \subset L_2',\ L_1 \in \mathscr{L}_1\} && (\text{ by (5)[a] })\end{aligned}$$

But $L_1 \subset L_2' \Rightarrow \mu^\wedge(L_1) \le \mu^\wedge(L_2')$, taking sup $\Rightarrow$

$$\sup\{\mu^\wedge(L_1) : L_1 \subset L_2',\ L_1 \in \mathscr{L}_1\} \le \mu^\wedge(L_2')$$

Hence,

$$\mu^\wedge(L_2') = \sup\{\mu^\wedge(L_1) : L_1 \subset L_2',\ L_1 \in \mathscr{L}_1\}$$

which means that $\mu^\wedge$ is $\mathscr{L}_1$*-regular on* $\mathscr{L}_2'$. Since $\mathscr{L}_1 \subset \mathscr{L}_2$, $\mu^\wedge$ is also $\mathscr{L}_2$*-regular on* $\mathscr{L}_2'$. Now any element of $A(\mathscr{L}_2)$ is, of the form

$$\bigcup_{i=1}^n (A_i \cap B_i') \qquad A_i, B_i \in \mathscr{L}_2$$

Consequently, $\mu^\wedge \in M_R(\mathscr{L}_2)$.

(7)[b] : Suppose $\mathscr{L}_2$ *semi-separates* $\mathscr{L}_1$.

Now $P \subset A_2' \cap L_2' \Rightarrow P \subset L_2' \Rightarrow P \cap L_2 = \emptyset$

and $P \in \mathscr{L}_1$, $L_2 \in \mathscr{L}_2$.

$\mathscr{L}_2$ *semi-separates* $\mathscr{L}_1 \Rightarrow$

$\exists\, \tilde{L}_2 \in \mathscr{L}_2$ s.t. $P \subset \tilde{L}_2 \subset L_2'$. $\therefore P' \supset \tilde{L}_2' \supset L_2 \Rightarrow A_2' \cap P' \supset A_2' \cap \tilde{L}_2'$

From [v],

$$\begin{aligned}\mu^\wedge(A_2') &\ge \mu(P) + \mu_*(A_2' \cap P')\\ &\ge \mu(P) + \mu_*(A_2' \cap \tilde{L}_2')\\ &= \mu(P) + \mu^\wedge(A_2' \cap \tilde{L}_2') && (\text{ by (4) })\\ &\ge \mu(P) + \mu^\wedge(A_2' \cap L_2) && (\tilde{L}_2' \supset L_2)\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\mu^\wedge(A_2') &\ge \sup\{\mu(P) : A_2' \cap L_2' \supset P \in \mathscr{L}_1\} + \mu^\wedge(A_2' \cap L_2)\\ &= \mu_*(A_2' \cap L_2') + \mu^\wedge(A_2' \cap L_2)\\ &= \mu^\wedge(A_2' \cap L_2') + \mu^\wedge(A_2' \cap L_2), \quad \forall A_2 \in \mathscr{L}_2\ (\mu^\wedge = \mu_* \text{ on } \mathscr{L}_2')\end{aligned}$$

We conclude, from (6), that every element of $\mathscr{L}_2'$ is $\mu^\wedge$-measurable. $\therefore A(\mathscr{L}_2) = A(\mathscr{L}_2') \subset \{\mu^\wedge\text{-measurable sets}\}$. By a standard Carathéodory argument, $\mu^\wedge|_{\mathscr{L}_2}$ is a *finitely additive measure on* $A(\mathscr{L}_2)$.

Let $L_2 \in \mathscr{L}_2$. Suppose $L_1 \subset L_2'$ $L_1 \in \mathscr{L}_1$. $\mathscr{L}_2$ *semi-separates* $\mathscr{L}_1$

$\Rightarrow \exists\, \tilde{L}_2 \in \mathscr{L}_2$ s.t. $L_1 \subset \tilde{L}_2 \subset L_2'$ and $\mu^\wedge = \mu$ on $\mathscr{L}_1$,

$$\begin{aligned}\mu(L_1) &= \mu^\wedge(L_1)\\ &\le \mu^\wedge(\tilde{L}_2) \le \mu^\wedge(L_2')\end{aligned}$$

$\therefore$

$$\mu(L_1) \le \mu^\wedge(\tilde{L}_2) \le \mu^\wedge(L_2')$$

Taking sup,

$$\sup\{\mu(L_1) : L_1 \subset L_2',\ L_1 \in \mathscr{L}_1\} \le \sup\{\mu^\wedge(\tilde{L}_2) : \tilde{L}_2 \subset L_2',\ \tilde{L}_2 \in \mathscr{L}_2\} \le \mu^\wedge(L_2')$$

But $\mu^\wedge(L_2') = \sup\{\mu(L_1) : L_1 \subset L_2',\ L_1 \in \mathscr{L}_1\}$ Hence,

$\mu^\wedge(L_2') = \sup\{\mu(L_1) : L_1 \subset \hat{L}_2,\ L_1 \in \mathscr{L}_1\} = \sup\{\mu^\wedge(\hat{L}_2) : \hat{L}_2 \subset L_2',\ \hat{L}_2 \in \mathscr{L}_2\}$

$\therefore$ $\mu^\wedge$ is $\mathscr{L}_1$-*regular on* $\mathscr{L}_2'$ and $\mathscr{L}_2$-*regular on* $\mathscr{L}_2'$, and consequently, $\mu^\wedge \in M_R(\mathscr{L}_2)$.

**Note** : If $\mathscr{L}_1 \subset \mathscr{L}_2$, then $\mathscr{L}_2$ trivially semi-separates $\mathscr{L}_1$, (7)[a] $\Rightarrow$ (7)[b]. ∎

## Corollary 3.1

Suppose $\mathscr{L}_1 = \mathscr{L}_2 = \mathscr{L}$, $X \in \mathscr{L}$, and $\mathscr{L}$ *coallocates itself*, ($\mathscr{L}$ is *normal*). then

(1) $\mu^\wedge$ is *finitely additive* and $\mu^\wedge(X) = \mu_\bullet(X) = \mu(X)$

(2) $\mu^\wedge(L) + \mu_\bullet(L') = \mu(X) \quad \forall L \in \mathscr{L}$

*Proof*:

(1) Direct consequences of Theorem 3.1.

(2) From Theorem 3.1(4), $\mu^\wedge = \mu_\bullet$ on $\mathscr{L}_2' = \mathscr{L}'$. $\therefore$ $\mu^\wedge(L') = \mu_\bullet(L')$ $\forall L \in \mathscr{L}$

Now $\mu^\wedge$ is finitely additive,

$$\mu^\wedge(X) = \mu^\wedge(L \cup L') = \mu^\wedge(L) + \mu^\wedge(L') = \mu^\wedge(L) + \mu_\bullet(L')$$

But $\mu^\wedge(X) = \mu_\bullet(X) = \mu(X)$, $\therefore$ $\mu^\wedge(L) + \mu_\bullet(L') = \mu(X)$. ∎

The coallocation theorem leads to the following direct consequences whose proofs are omitted.

**THEOREM 3.2** [*Regular measure extension on coallocation lattices*]

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$ and $\mu \in M_R(\mathscr{L}_1)$. If $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$, then there *exists* a *unique* $\nu \in M_R(\mathscr{L}_2)$, s.t. on $\mathscr{L}_1$, $\mu = \nu|_{\mathscr{L}_1} \in M_R(\mathscr{L}_1)$. Furthermore, $\nu$ is $\mathscr{L}_1$-*regular on* $\mathscr{L}_2'$. Note that $\nu = \mu^\wedge|_{\mathscr{L}_2}$.

**THEOREM 3.3** [*Regular measure enlargement on coallocation lattices*]

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$ and $\mu \in M(\mathscr{L}_1)$. If $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$, then $\exists\ \nu \in M_R(\mathscr{L}_2)$, s.t. $\mu \le \nu$ on $\mathscr{L}_1$ and $\mu(X) = \nu(X)$.

**THEOREM 3.4** [*Regular measure enlargement on a normal lattice*]

Suppose $\mathscr{L}$ is *normal* and $\mu \in M(\mathscr{L})$. Then there exists a unique $\nu \in M_R(\mathscr{L})$, s.t. $\mu \le \nu$ on $\mathscr{L}$ and $\mu(X) = \nu(X)$.

Furthermore, if we impose a $\sigma$-smoothness condition on $\mu$, we obtain the following :

**THEOREM 3.5**

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$, and $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$, and $\mu \in M_R^\sigma(\mathscr{L}_1)$,

$\nu \in M_R(\mathscr{L}_2)$, where $\nu$ is the regular measure extension of $\mu$. Then

$$\nu \in M_R(\mathscr{L}_2) \cap M_\sigma(\mathscr{L}_2').$$

*Proof*:

$\mu \in M_R^\sigma(\mathscr{L}_1) \Rightarrow \mu \in M_R(\mathscr{L}_1)$ hence by Theorem 3.2,

$\nu \equiv \mu^\wedge|_{\mathscr{L}_2} \in M_R(\mathscr{L}_2)$ is the unique regular measure extension of $\mu$.

Theorem 3.1(7) $\Rightarrow \mu^\wedge$ is $\mathscr{L}_1$-regular on $\mathscr{L}_2'$.

$\Rightarrow \forall\ B_n' \in \mathscr{L}_2',\ B_n' \downarrow \emptyset,\ \exists\ A_n \in \mathscr{L}_1,\ A_n \subset B_n',\ \forall \varepsilon > 0,$

$\mu^\wedge(B_n') < \mu^\wedge(A_n) + \varepsilon/2$

$\mu^\wedge = \mu$ on $\mathscr{L}_1$, since $\mu$ is $\mathscr{L}_1$-regular [Theorem 3.1(5)[b]]

$\mu^\wedge(B_n') < \mu(A_n) + \varepsilon/2$ ..... [i]

Since $A_n \subset B_n' \downarrow \emptyset$, without the loss of generality, we may assume $A_n \downarrow \emptyset$. Thus, $\mu(A_n) \to 0$ $\because \mu$ is $\sigma$-smooth on $\mathscr{L}_1$

Hence, [i] $\Rightarrow$ $\mu^\wedge(B_n') < \varepsilon$

Consequently, $\mu^\wedge(B_n') \to 0$ $\forall\ B_n' \in \mathscr{L}_2'$

Or, $\nu = \mu^\wedge|_{\mathscr{L}_2} \in M_R(\mathscr{L}_2) \cap M_\sigma(\mathscr{L}_2')$. ∎

In particular, if $\mathscr{L}_1 = \mathscr{L}_2 = \mathscr{L}$ is normal, we have

## Corollary 3.5

Suppose $\mathscr{L}$ is *normal*, and $\mu \in M_\sigma(\mathscr{L})$, $\nu \in M_R(\mathscr{L})$, $\nu$ is the regular measure enlargement of $\mu$, $\mu \le \nu$ on $\mathscr{L}$, $\mu(X) = \nu(X)$. Then, $\nu \in M_\sigma(\mathscr{L}')$.

## Theorem 3.6

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$, and $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$, and $\mathscr{L}_2$ is *countably paracompact* and *normal*. Suppose $\mu \in M_R^\sigma(\mathscr{L}_1)$, and $\nu \in M_R(\mathscr{L}_2)$, where $\nu$ is the unique regular measure extension of $\mu$. Then, $\nu \in M_R^\sigma(\mathscr{L}_2)$.

*Proof*:

$\mathscr{L}_2$ c.p. $\Rightarrow \forall\ B_n \in \mathscr{L}_2,\ B_n \downarrow \emptyset,\ \exists\ \tilde{B}_n \in \mathscr{L}_2,\ B_n \subset \tilde{B}_n' \downarrow \emptyset$

Theorem 3.5 $\Rightarrow \nu \in M_\sigma(\mathscr{L}_2'),\ \nu(\tilde{B}_n') \to 0 \Rightarrow \nu(B_n) \to 0 \quad \forall\ B_n \in \mathscr{L}_2$

$\therefore \nu$ is $\sigma$-smooth on $\mathscr{L}_2$, and since $\nu$ is regular on $\mathscr{L}_2$, $\nu \in M_R^\sigma(\mathscr{L}_2)$. ∎

We now give two applications of the results on coallocation lattices to topological spaces.

**1) MEASURES ON A LOCALLY COMPACT HAUSDROFF SPACE**

Let $X$ be a *locally compact Hausdroff space* and $\mathscr{L}_1 = K_0$ is the collection of all *compact* $G_\delta$-sets, while $\mathscr{L}_2 = K$ is the collection of all compact sets. Note that in this case, $X$ does not belong to either $K_0$ or $K$, unless $X$ is compact. Then $K_0 \subset K$, and it can be shown that $K$ coallocates $K_0$. For any $\mu \in M_R(K_0)$, $\mu$ is $\sigma$-smooth, because $K_0$ is compact. Thus, $\mu \in M_R^\sigma(K_0)$. By the coallocation theorem, we can extend $\mu$ uniquely to a regular measure $\nu$ which is also $\sigma$-smooth, because $K$ is compact. Hence, $\nu \in M_R^\sigma(K)$.

**2) MEASURE ENLARGEMENT FROM ZERO SETS TO CLOSED SETS**

Suppose $X$ is a *countably paracompact* and *normal* topological space. Let $\mathscr{L}_1 = \mathfrak{Z}$ (*zero* sets) and $\mathscr{L}_2 = \mathfrak{S}$ (*closed* sets). That is, $\mathfrak{S}$ is c.p. normal. $\mathfrak{Z} \subset \mathfrak{S}$ because all zero sets are closed $G_\delta$-sets, and disjoint closed sets can be separated by disjoint zero sets. Therefore, $\mathfrak{Z}$ is c.p. and normal. Thus, $\mathfrak{Z}$ coseparates $\mathfrak{S}$. Hence $\mathfrak{S}$ coallocates $\mathfrak{Z}$ [Theorem 2.3].

Let $\mu \in M_R(\mathfrak{Z})$. Then by Theorem 3.2, there exists a unique regular measure extension $\nu \in M_R(\mathfrak{S})$. Theorem 3.1(7)[a] implies $\nu$ on all open sets is $\mathfrak{Z}$-regular.

Suppose $\mu \in M_R^\sigma(\mathfrak{Z})$. By Theorem 3.5, the unique regular measure extension is $\nu \in M_R(\mathfrak{S}) \cap M_\sigma(\mathfrak{S}')$. Now $\mathfrak{S}$ is c.p., hence $\nu \in M_\sigma(\mathfrak{S})$ [Theorem 2.4]. Then, $\nu \in M_R^\sigma(\mathfrak{S})$. This is the result of Marik [5].

# 4. NORMAL LATTICES

In this section, we give further characterization of normal lattices and further consequences of a lattice being normal in terms of associated measures on the generalized algebra.

**Definition 4.1**

Let $\mathscr{L}$ be a lattice of subsets of $X$, and $\mu \in M(\mathscr{L})$. $\forall E \subset X$, define

$$\mu'(E) \equiv \inf \{ \mu(\tilde{L}') : E \subset \tilde{L}', \tilde{L} \in \mathscr{L} \}$$

$$\mu''(E) \equiv \inf \{ \textstyle\sum_{n=1}^\infty \mu(\tilde{L}_n') : E \subset \bigcup_{n=1}^\infty \tilde{L}_n', \tilde{L}_n \in \mathscr{L} \}$$

$$\mu_*(E) \equiv \sup \{ \mu(\tilde{L}) : E \supset \tilde{L} \in \mathscr{L} \}$$

$$\mu^\wedge(E) \equiv \inf \{ \mu_*(\tilde{L}') : E \subset \tilde{L}', \tilde{L} \in \mathscr{L} \}$$

It is clear that if $\mu \in M_R(\mathscr{L})$, then $\mu = \mu'$ on $A(\mathscr{L})$.

**THEOREM 4.1**

Let $\mu,\nu \in M(\mathscr{L})$, such that $\mu(X) = \nu(X)$. Then

$$\mu \le \nu \text{ on } \mathscr{L} \quad \Leftrightarrow \quad \mu \le \nu \le \nu' \le \mu' \text{ on } \mathscr{L}$$

*Proof*:

It is obvious that $\mu \le \nu$ on $\mathscr{L} \Leftrightarrow \nu \le \mu$ on $\mathscr{L}'$. Let $E \subset X$ s.t. $E \subset \tilde{L}'$, $\tilde{L} \in \mathscr{L}$, $\nu(\tilde{L}') \le \mu(\tilde{L}')$. Taking inf,

$\inf\{ \nu(\tilde{L}') : \tilde{L}' \in \mathscr{L}' \} \le \inf\{ \mu(\tilde{L}') : \tilde{L}' \in \mathscr{L}' \}$,

$\therefore \nu'(E) \le \mu'(E)$. In particular, $E \in \mathscr{L} \Rightarrow \nu' \le \mu'$ on $\mathscr{L}$. Hence, $\mu \le \nu \le \nu' \le \mu'$ on $\mathscr{L}$. ∎

**THEOREM 4.2**

Suppose $\forall \mu \in I(\mathscr{L})$, $L_1, L_2 \in \mathscr{L}$,

$$\mu'(L_1) = 1 \text{ and } \mu'(L_2) = 1 \quad \Rightarrow \quad \mu'(L_1 \cap L_2) = 1$$

Then, $\mathscr{L}$ is *normal*.

*Proof*:

Suppose $\mathscr{L}$ is not normal. Then $\exists \mu \in I(\mathscr{L})$, $\nu_1, \nu_2 \in I_R(\mathscr{L})$, s.t. $\mu \le \nu_1$ on $\mathscr{L}$, $\mu \le \nu_2$ on $\mathscr{L}$, but $\nu_1 \ne \nu_2$.

$\therefore \exists L_1, L_2 \in \mathscr{L}$, $L_1 \cap L_2 = \emptyset$,

$\nu_1(L_1) = 1$, $\nu_2(L_1) = 0$ and $\nu_1(L_2) = 0$, $\nu_2(L_2) = 1$

Now if $L_1 \subset \tilde{L}_1'$, $\tilde{L}_1 \in \mathscr{L}$, then $\nu_1(\tilde{L}_1') = 1$. Since $\mu \le \nu_1$ on $\mathscr{L} \Leftrightarrow \nu_1 \le \mu$ on $\mathscr{L}'$, we have $\mu(\tilde{L}_1') = 1 \Rightarrow \mu'(L_1) = 1$.

Similarly, if $L_2 \subset \tilde{L}_2'$, $\tilde{L}_2 \in \mathscr{L}$, then $\mu'(L_2) = 1$.

Then, by assumption, $\mu'(L_1 \cap L_2) = 1$. But $L_1 \cap L_2 = \emptyset$, $\therefore \mu'(L_1 \cap L_2) = 0$ gives a contradiction. Consequently, $\mathscr{L}$ is normal. ∎

**THEOREM 4.3**

Let $\nu \in M_R(\mathscr{L})$, $\rho \in M(\mathscr{L})$, s.t. $\nu(X) = \rho(X)$ and on $\mathscr{L}'$, $\nu \le \rho \in M_R(\mathscr{L}')$. Then

(1) $\rho \le \nu = \nu' \le \rho'$ on $\mathscr{L}$

(2) $\mathscr{L}$ is *normal* $\Rightarrow$ $\nu = \nu' = \rho'$ on $\mathscr{L}$.

*Proof*:

(1) $\nu \le \rho$ on $\mathscr{L}' \Leftrightarrow \rho \le \nu$ on $\mathscr{L}$, and $\nu \in M_R(\mathscr{L}) \Rightarrow \nu = \nu'$. Hence by Theorem 4.1, $\rho \le \nu = \nu' \le \rho'$ on $\mathscr{L}$.

(2) Suppose $\mathscr{L}$ is normal and $\exists L \in \mathscr{L}$ s.t. $\nu(L) < \rho'(L)$.

$\nu \in M_R(\mathscr{L}) \Rightarrow \forall \varepsilon > 0, \exists \tilde{L} \in \mathscr{L}$, $\tilde{L} \subset L'$, $\nu(\tilde{L}) + \varepsilon > \nu(L')$

$\therefore \nu(\tilde{L}') < \nu(L) + \varepsilon$ and $L \cap \tilde{L} = \emptyset$

By normality, $\exists L_a, L_b \in \mathscr{L}$, s.t. $L \subset L_a'$, $\tilde{L} \subset L_b'$, $L_a' \cap L_b' = \emptyset$

$\therefore \quad L \subset L_a' \subset L_b \subset \tilde{L}'$

$\nu(L) < \rho'(L) \leq \rho'(L_a') \underset{\mathscr{L}'}{=} \rho(L_a') \leq \rho(L_b) \underset{\mathscr{L}}{\leq} \nu(L_b) \leq \nu(\tilde{L}') < \nu(L) + \varepsilon$

$\Rightarrow \rho'(L) \leq \nu(L)$ gives a contradiction. $\therefore \nu = \nu' = \rho'$ on $\mathscr{L}$. ∎

**THEOREM 4.4**

Let $\mathscr{L}$ be a lattice of subsets of $X$, and $\mu \in M_\sigma(\mathscr{L})$. Then,

(1) $\mu'' \leq \mu'$ *everywhere*

(2) $\mu' = \mu$ on $\mathscr{L}'$

(3) $\mu \leq \mu'' \leq \mu'$ on $\mathscr{L}$

(4) $\mu(X) = \mu''(X) = \mu'(X)$

(5) $\mu_*(L') + \mu'(L) = \mu(X), \quad \forall L \in \mathscr{L}$

(6) If $\mathscr{L}$ is *normal*, then $\mu \leq \mu'' \leq \mu' = \mu^\wedge$ *on* $\mathscr{L}$

(7) If $\mathscr{L}$ is *δ-normal*, then $\mu'' = \mu'$ *on* $\mathscr{L}$.

**NOTE :** The condition $\mu \in M_\sigma(\mathscr{L})$ is imposed, because when $\mu$ is a 0-1 measure and if $\mu$ is not σ-smooth, then $\mu'' \equiv 0$.

*Proof*:

(1) By definition of $\mu''$, the inf encompasses more sets than that of $\mu'$, hence $\mu'' \leq \mu'$ everywhere.

(2) Take $E = \tilde{L}' \in \mathscr{L}'$, $\therefore \mu' = \mu$ on $\mathscr{L}'$. In particular,

$$\mu'(X) = \mu(X) \qquad \text{..... [i]}$$

(3) From (1) and [i], we have $\mu''(X) \leq \mu(X)$. We now show that $\mu''(X) = \mu(X)$. For suppose $X = \bigcup_{i=1}^\infty L_i'$, pairwise disjoint $L_i' \in \mathscr{L}'$, and $\sum_{i=1}^\infty \mu(L_i') < \mu(X)$, but $\sum_{i=1}^\infty \mu(L_i') = \lim_{n\to\infty} \sum_{i=1}^n \mu(L_i')$ $\geq \lim_{n\to\infty} \mu(\bigcup_{i=1}^n L_i') = \mu(X)$.

Since $\bigcup_{i=1}^n L_i' \in \mathscr{L}'$ and $\bigcup_{i=1}^n L_i' \uparrow X$, or $\bigcap_{i=1}^\infty L_i \downarrow \emptyset$, also $\mu \in M_\sigma(\mathscr{L})$.

Taking the inf of the above, we have $\mu''(X) \geq \mu(X)$.

Consequently, $\mu''(X) = \mu(X)$ ..... [ii]

Now suppose $\exists\, L \in \mathscr{L}$, $\mu(L) > \mu''(L)$,

$$\begin{aligned} \mu''(X) = \mu''(L \cup L') &\leq \mu''(L) + \mu''(L') \\ &\leq \mu''(L) + \mu(L') && (\because \mu'' \leq \mu \text{ on } \mathscr{L}') \\ &< \mu(L) + \mu(L') && (\text{ by assumption}) \\ &= \mu(X) && \text{contradicting [ii]} \end{aligned}$$

$\therefore \mu \leq \mu''$ on $\mathscr{L}$. Together with (1), we have $\mu \leq \mu'' \leq \mu'$ on $\mathscr{L}$.

(4) [i] and [ii] $\Rightarrow \mu(X) = \mu''(X) = \mu'(X)$.

(5) $\forall \varepsilon > 0, \quad \mu_*(L') - \varepsilon < \mu(\hat{L}) \quad$ where $L' \supset \hat{L} \in \mathscr{L}$

$\therefore \qquad \mu(X) - \mu_*(L') > \mu(\hat{L}') - \varepsilon$

Taking inf, we have

$$\mu(X) - \mu_*(L') \geq \mu'(L) \quad \text{..... [iii]}$$

If $L' \supset \hat{L}$, $\mu_*(L') \geq \mu(L') \geq \mu(\hat{L})$ since $\mu_* \leq \mu$ on $\mathscr{L}'$

$\therefore \quad \mu(X) - \mu_*(L') \leq \mu(X) - \mu(\hat{L}) = \mu(\hat{L}')$

Taking inf, we have

$$\mu(X) - \mu_*(L') \leq \inf\{\, \mu(\hat{L}') : L \subset \hat{L}',\ \hat{L} \in \mathscr{L} \,\} \equiv \mu'(L) \quad \text{..... [iv]}$$

[iii] and [iv] $\Rightarrow \mu(X) - \mu_*(L') = \mu'(L) \quad \forall L \in \mathscr{L}$.

(6) L is normal, then Corollary 3.1(2) and (5) imply $\mu^\wedge(L) = \mu'(L) \quad \forall L \in \mathscr{L}$.

Now $\mu' = \mu^\wedge$ on $\mathscr{L}$, and from (3), $\mu \leq \mu'' \leq \mu' = \mu^\wedge$ on $\mathscr{L}$.

(7) From (1), $\mu'' \leq \mu'$ everywhere. Suppose $\exists L \in \mathscr{L}$, s.t. $\mu''(L) < \mu'(L)$

Then $L \subset \bigcup_{n=1}^{\infty} L_n'$, $L_n \in \mathscr{L}$. $\mathscr{L}$ is $\delta \Rightarrow \bigcap_{n=1}^{\infty} L_n \in \mathscr{L}$.

Let $D \equiv \bigcap_{n=1}^{\infty} L_n$, $L \cap D = \emptyset$. $\mathscr{L}$ is normal $\Rightarrow \exists A, B \in \mathscr{L}$, s.t.

$$L \subset A',\ D \subset B',\ A' \cap B' = \emptyset \quad \therefore \quad L \subset A' \subset B \subset D' \equiv \bigcup_{n=1}^{\infty} L_n'$$

$$\mu'(L) \leq \mu'(A') \underset{\text{from (2)}}{=} \mu(A') \leq \mu(B) \underset{\text{from (3)}}{\leq} \mu''(B) \leq \sum_{n=1}^{\infty} \mu(L_n')$$

Taking inf, and since $L \subset \bigcup_{n=1}^{\infty} L_n'$,

$$\mu'(L) \leq \inf\{\, \textstyle\sum_{n=1}^{\infty} \mu(L_n') : L \subset \sum_{n=1}^{\infty} L_n',\ L_n \in \mathscr{L} \,\} \equiv \mu''(L)$$

gives a contradiction. $\therefore \mu''(L) = \mu'(L)$, or $\mu'' = \mu'$ on $\mathscr{L}$. ∎

**THEOREM 4.5**

Let $\mathscr{L}$ be a lattice of subsets of $X$, and let $\mu \in M_\sigma(\mathscr{L})$, $\rho \in M(\mathscr{L})$, s.t. $\mu \leq \rho$ on $\mathscr{L}$, $\mu(X) = \rho(X)$.

If $\mathscr{L}$ is *countably paracompact* and *normal*, then $\rho \in M_\sigma(\mathscr{L})$.

*Proof*:

Let $L_n \downarrow \emptyset$, $L_n \in \mathscr{L}$, $\forall n$

$\mathscr{L}$ c.p. $\Rightarrow \exists \tilde{L}_n \in \mathscr{L}$, $L_n \subset \tilde{L}_n' \downarrow \emptyset \quad \therefore L_n \cap \tilde{L}_n = \emptyset$

$\mathscr{L}$ normal $\Rightarrow \exists A_n, B_n \in \mathscr{L}$, $L_n \subset A_n'$, $\tilde{L}_n \subset B_n'$, $A_n' \cap B_n' = \emptyset$.

Or, $L_n \subset A_n' \subset B_n \subset \tilde{L}_n' \downarrow \emptyset$, $\therefore \rho(L_n) \leq \rho(A_n') \leq \mu(A_n') \leq \mu(B_n) \to 0$

(one may assume, with the loss of generality, $B_n \downarrow$).

($\because \rho \leq \mu$ on $\mathscr{L}'$; $B_n \downarrow \emptyset$ and $\mu \in M_\sigma(\mathscr{L})$) $\therefore \rho(L_n) \to 0$, or $\rho \in M_\sigma(\mathscr{L})$. ∎

**THEOREM 4.6**

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$, and $\mathscr{L}_1$ *separates* $\mathscr{L}_2$. Then,

$$\mathscr{L}_1 \textit{ normal} \Leftrightarrow \mathscr{L}_2 \textit{ normal}.$$

*Proof*:

"$\Rightarrow$" Suppose $\mathscr{L}_1$ is normal.

Let $\mu \in I(\mathscr{L}_2)$, $\nu_a, \nu_b \in I_R(\mathscr{L}_2)$, s.t. $\mu \le \nu_a$ on $\mathscr{L}_2$, $\mu \le \nu_b$ on $\mathscr{L}_2$.
Then $\mu| \in I(\mathscr{L}_1)$, $\nu_a|, \nu_b| \in I_R(\mathscr{L}_1)$, and $\mu| \le \nu_a|$ on $\mathscr{L}_1$, $\mu| \le \nu_b|$ on $\mathscr{L}_1$. $\mathscr{L}_1$ normal $\Leftrightarrow$ $\nu_a| = \nu_b|$ [Theorem 2.2].
Extend $\nu_a|$ and $\nu_b|$ to $\mathscr{L}_2 \Rightarrow$ $\nu_a = \nu_b \Leftrightarrow \mathscr{L}_2$ normal,
$\because$ $\mathscr{L}_1$ separates $\mathscr{L}_2$, the extension is unique.

"$\Leftarrow$" Suppose $\mathscr{L}_2$ is normal.
Let $\mu \in I(\mathscr{L}_1)$, $\nu_a, \nu_b \in I_R(\mathscr{L}_1)$, s.t. $\mu \le \nu_a$ on $\mathscr{L}_1$, $\mu \le \nu_b$ on $\mathscr{L}_1$.
Extend $\mu$ to $\lambda \in I(\mathscr{L}_2)$, and $\nu_a, \nu_b$ to $\tau_a, \tau_b \in I_R(\mathscr{L}_2)$, respectively. We now show that $\lambda \le \tau_a$ on $\mathscr{L}_2$, and $\lambda \le \tau_b$ on $\mathscr{L}_2$.
For suppose $\exists$ $L_2 \in \mathscr{L}_2$ s.t. $\lambda(L_2) = 1$ but $\tau_a(L_2) = 0$. Then $\tau_a(L_2') = 1$. But $\tau_a \in I_R(\mathscr{L}_2)$, $\exists$ $\tilde{L}_2 \in \mathscr{L}_2$, s.t. $\tilde{L}_2 \subset L_2'$, $\tau_a(\tilde{L}_2) = 1$
Since $\mathscr{L}_1$ separates $\mathscr{L}_2 \Rightarrow \exists$ $L_1 \in \mathscr{L}_1$, s.t. $L_2 \subset L_1 \subset \tilde{L}_2'$

$$\therefore\ 1 = \lambda(L_2) \le \lambda(L_1) \underset{\mathscr{L}_1}{=} \mu(L_1) \underset{\mathscr{L}_1}{\le} \nu_a(L_1) \underset{\mathscr{L}_1}{=} \tau_a(L_1) \le \tau_a(\tilde{L}_2')$$

Thus, $\tau_a(\tilde{L}_2') = 1$ or $\tau_a(\tilde{L}_2) = 0$ contradicting $\tau_a(\tilde{L}_2) = 1$
$\therefore$ $\lambda \le \tau_a$ on $\mathscr{L}_2$. Similarly, $\lambda \le \tau_b$ on $\mathscr{L}_2$. Since $\mathscr{L}_2$ is normal, $\tau_a = \tau_b$ $\therefore$ $\tau_a| = \tau_b|$, i.e. $\nu_a = \nu_b \Leftrightarrow \mathscr{L}_1$ is normal. ∎

**THEOREM 4.7**

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$, and $\mu \in M_R(\mathscr{L}_1)$, $\nu \in M_R(\mathscr{L}_2)$, s.t. $\mu(X) = \nu(X)$, $\nu|_{\mathscr{L}_1} = \mu$. Then

$\mathscr{L}_1$ *separates* $\mathscr{L}_2 \Rightarrow$ $\nu$ is $\mathscr{L}_1$*-regular on* $\mathscr{L}_2'$.

*Proof*:

$\nu \in M_R(\mathscr{L}_2)$, $\therefore$ $\forall$ $L_2' \in \mathscr{L}_2'$, $\nu(L_2') = \sup\{\ \nu(\tilde{L}_2)\ :\ L_2' \supset \tilde{L}_2 \in \mathscr{L}_2\ \}$
$\forall \varepsilon > 0$, $L_2' \supset \tilde{L}_2 \in \mathscr{L}_2$, $\nu(L_2') < \nu(\tilde{L}_2) + \varepsilon$
$L_2 \cap \tilde{L}_2 = \emptyset$, and $\mathscr{L}_1$ separates $\mathscr{L}_2 \Rightarrow$
$\exists$ $L_1, \tilde{L}_1 \in \mathscr{L}_1$, s.t. $L_2 \subset L_1$, $\tilde{L}_2 \subset \tilde{L}_1$, $L_1 \cap \tilde{L}_1 = \emptyset$

$$\begin{aligned} \nu(L_2') &< \nu(\tilde{L}_2) + \varepsilon \\ &\le \nu(\tilde{L}_1) + \varepsilon \quad (\ \tilde{L}_2 \subset \tilde{L}_1\ ) \\ &= \mu(\tilde{L}_1) + \varepsilon \quad (\ \nu|_{\mathscr{L}_1} = \mu\ ) \end{aligned}$$

Taking sup, $\nu(L_2') = \sup\{\ \mu(\tilde{L}_1)\ :\ L_2' \supset \tilde{L}_1 \in \mathscr{L}_1\ \}$, $\forall$ $L_2' \in \mathscr{L}_2'$
i.e. $\nu$ is $\mathscr{L}_1$*-regular on* $\mathscr{L}_2'$. ∎

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