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A NOTE ON THE FREE CONVECTION BOUNDARY LAYER ON A VERTICAL SURFACE WITH PRESCRIBED HEAT FLUX AT SMALL PRANDTL NUMBER

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ABSTRACT. It is shown that for a particular case of the surface heat flux the equations for small Prandtl number have simple analytical solutions. These are presented and compared with numerical solutions of the general equations.

KEY WORDS AND PHRASES. Free convection, boundary layers and heat flux.
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1. INTRODUCTION.

In a recent paper [1], the solution for the free convection boundary-layer flow on a vertical plate with a prescribed surface heat flux valid for small Prandtl numbers was derived. The surface heat flux was taken to be proportional to x^λ, where x is the distance from the leading edge and λ is a constant, with the governing equations then being reducible to similarity form. Results in [1] were given for the case of uniform wall heat flux, i.e., $\lambda = 0$. A further consideration of this problem reveals that, for the case when $\lambda = 1$, simple analytical solutions are possible. It is the purpose of this note to present these solutions, and, as analytical solutions in free convection boundary-layer theory are somewhat of a rarity, this analysis is worth describing.

2. ANALYSIS.

Following [1], the governing similarity equations are, for $\lambda = 1$,

$$f''' + \theta + ff'' - f'^2 = 0 \tag{2.1a}$$

$$\theta'' + \sigma(f\theta' - f'\theta) = 0 \tag{2.1b}$$

with,

$$f(0) = 0, \quad f'(0) = 0, \quad \theta'(0) = -1, \quad f' \to 0, \quad \theta \to 0 \text{ as } \eta \to \infty\,, \tag{2.1c}$$

where primes denote differentiation with respect to the independent variable η and σ is the Prandtl number. There is an inner region, in which

$$f = \sigma^{-1/10}F(\zeta), \quad \theta = \sigma^{-2/5}H(\zeta), \quad \zeta = \sigma^{-1/10}\eta\,. \tag{2.2}$$

A consideration of the equations in this region leads to,

$$H = a_0 + \sigma^{1/2}(a_1 - \zeta) + \dots . \tag{2.3a}$$

The equation for F is, at leading order, given by a Falkner-Skan equation, and, as $\zeta \to \infty$,

$$F \sim a_0^{1/2}\zeta + b_0 + \sigma^{1/2}\left[-\frac{\zeta^2}{2a_0^{1/2}} + \frac{1}{2a_0}(a_0 a_1 - b_0)\zeta + b_1 \right] + \dots . \tag{2.3b}$$

The constants a_0 and a_1 are determined from the matching with the outer region, and b_0 is determined from the solution of the equation for F in the inner region. In the outer region,

$$f = \sigma^{-3/5}\phi(Y), \quad \theta = \sigma^{-2/5}h(Y), \quad Y = \sigma^{2/5}\eta . \tag{2.4}$$

Using (2.4) in equations (2.1abc) gives the equations for the outer region as

$$h + \phi\phi'' - \phi'^2 + \sigma\phi''' = 0 \tag{2.5a}$$

$$h'' + \phi h' - \phi' h = 0 \tag{2.5b}$$

(where primes now denote differentiation with respect to Y). The boundary conditions to be satisfied by equations (2.5ab) are that,

$$\phi' \to 0, \quad h \to 0 \quad \text{as } Y \to \infty \tag{2.6a}$$

and, from matching within the inner region, that

$$h \sim a_0 - Y + \dots + \sigma^{1/2}(a_1 + \dots) + \dots , \tag{2.6b}$$

$$\phi \sim a_0^{1/2} Y - \tfrac{1}{2} a_0^{-1/2} Y^2 + \dots + \sigma^{1/2}(b_0 + \frac{1}{2a_0}(a_0 a_1 - b_0)Y + \dots) + \dots \tag{2.6c}$$

for Y small.

(2.6bc) suggests looking for a solution of equations (2.5ab) by expanding

$$\phi = \phi_0 + \sigma^{1/2}\phi_1 + \dots, \qquad h = h_0 + \sigma^{1/2}h_1 + \dots \tag{2.7}$$

At leading order we obtain the equations

$$h_0 + \phi_0\phi_0'' - \phi_0'^2 = 0 \tag{2.8a}$$

$$h_0'' + \phi_0 h_0' - \phi_0' h_0 = 0 \tag{2.8b}$$

It is straightforward to show that the solution of equations (2.8ab), which satisfies boundary conditions (2.6abc) is

$$a_0 = 1, \quad \phi_0 = 1 - e^{-Y}, \quad h_0 = e^{-Y} . \tag{2.9}$$

The solution can be continued to higher order terms. We find that, at $0(\sigma^{1/2})$,

$$a_1 = -b_0, \quad \phi_1 = b_0 e^{-Y}, \quad h_1 = -b_0 e^{-Y} \tag{2.10}$$

Using the value for a_0 given by (2.9), the appropriate Falkner-Skan equation for the leading order term F_0 in the inner layer can be solved. This gives, [2],

$$F_0''(0) = 1.23259, \qquad b_0 = -0.64790 .$$

Then using the value for b_0, the (linear) equation for F_1, the term of $0(\sigma^{1/2})$ in the inner region, can be solved, giving

$$F_1''(0) = -0.41392, \qquad b_1 = -0.62264 .$$

3. RESULTS.

The analysis presented above gives, from (2.2), (2.3ab), (2.4), (2.9) and (2.10)

$$\left[\frac{d^2 f}{d\eta^2}\right]_{\eta=0} = \sigma^{-1/10}(1.23259 - 0.41392\,\sigma^{1/2} + 0(\sigma)) \tag{3.1a}$$

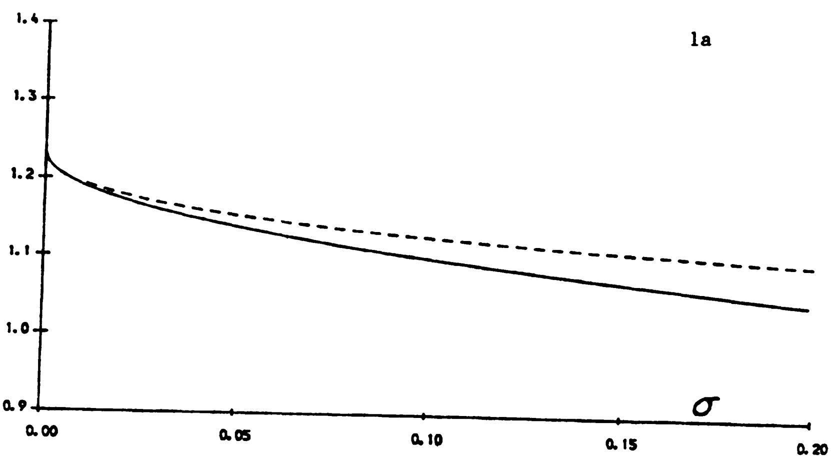
$$\theta(0) = \sigma^{-25}(1 + 0.64790\,\sigma^{1/2} + 0(\sigma)) \tag{3.1b}$$

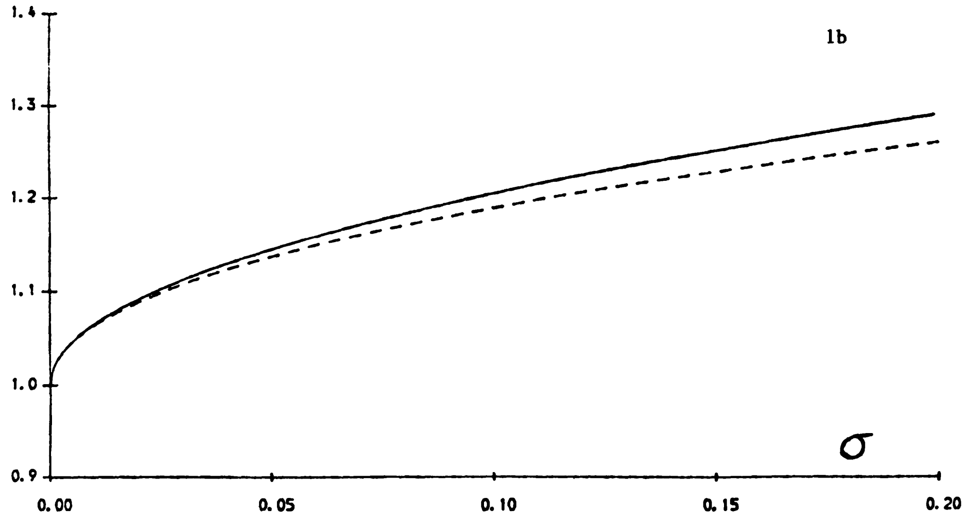
$$f(\infty) = \sigma^{-3/5}(1 + 0(\sigma)) \tag{3.1c}$$

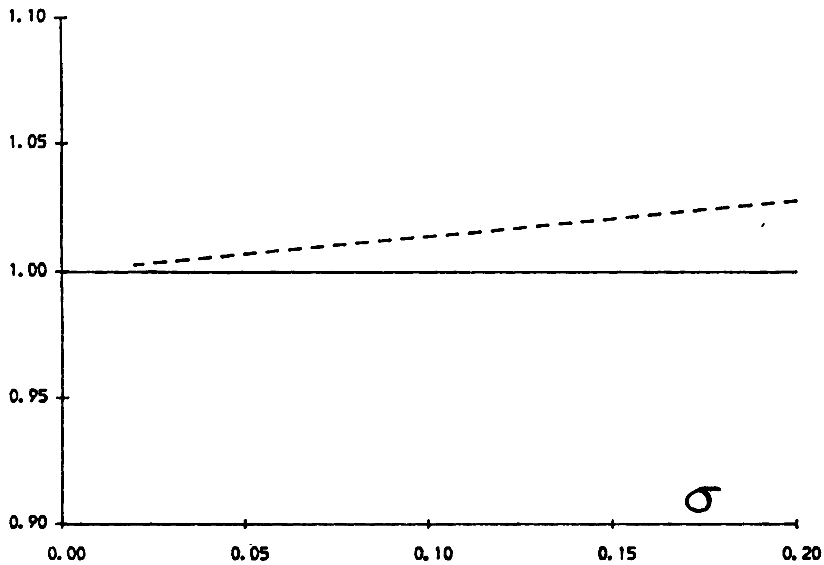
for σ small.

To check on the validity of the series approximations (3.1abc), we compared these with values obtained from a numerical solution of equations (2.1abc). The results are shown in figures 1, where we give the numerically determined values of $\left[\frac{d^2 f}{d\eta^2}\right]_0 \sigma^{1/10}$, $\theta(0)\sigma^{2/5}$ and $f(\infty)\sigma^{3/5}$ (shown by the broken line) and these quantities as calculated from (3.1abc) (shown by the full line). In all three cases we can see that the numerically determined values and (3.1abc) are in good agreement, even at the relatively large value of $\sigma = 0.2$, and that the agreement between the two sets of results improves as σ is decreased. It is worth noting that the linear slope of the numerical results in figure 1c appears to suggest that the correction to (3.1abc) is of $0(\sigma)$ and that no extra powers of σ are required (at least up to this order) in the expansions in the inner and outer regions (as was required in the general case given in [1]).

Figure 1 Graphs of (a) $\sigma^{1/10}\left[\frac{d^2 f}{d\eta^2}\right]_0$, (b) $\sigma^{2/5}\theta(0)$, and (c) $\sigma^{3/5}f(\infty)$ obtained from a numerical solution of equations (1) (broken line) and from series expansions (11) (full line).







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