Internat. J. Math. & Math. Sci.
VOL. 16 NO. 4 (1993) 687-694

OUTER MEASURES AND ASSOCIATED LATTICE PROPERTIES

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(Received January 30, 1992 and in revised form May 18, 1992)

ABSTRACT. Zero-one measure characterizations of lattice properties such as normality are extended to more general measures. For a given measure, we consider two associated "outer" measures and attempt to obtain the "outer"-measurable sets. We also seek necessary and sufficient conditions for the measure and outer measures to be equal on the lattice or its complement.

KEY WORDS AND PHRASES. Measure, lattice, outer-measure, measurable, normal, regular, almost countably compact, separates, semiseparates, support.
1991 AMS SUBJECT CLASSIFICATION CODES. 28C15, 28A12.

1. INTRODUCTION.

Let X be an arbitrary set and $\mathcal{L}$ a lattice of subsets of X. $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$, and $M(\mathcal{L})$ those bounded and finitely additive measures on $\mathcal{A}(\mathcal{L})$; $M_R(\mathcal{L})$ denotes those $\mu \in M(\mathcal{L})$ which are $\mathcal{L}$-regular, while $M_\sigma(\mathcal{L})$ denotes those $\mu \in M(\mathcal{L})$ which are σ-smooth on $\mathcal{L}$. Finally $I(\mathcal{L})$, $I_R(\mathcal{L})$, and $I_\sigma(\mathcal{L})$ are the nontrivial zero-one elements of $M(\mathcal{L})$, $M_R(\mathcal{L})$, and $M_\sigma(\mathcal{L})$ respectively.

Many well-known lattice properties can be completely characterized in terms of $I(\mathcal{L})$, $I_R(\mathcal{L})$ or $I_\sigma(\mathcal{L})$, e.g., normal lattices, regular lattices, disjunctive lattices, etc. (see [4], [5], [3], [10], [2]). We begin by extending many of these results to $M(\mathcal{L})$, $M_R(\mathcal{L})$, and $M_\sigma(\mathcal{L})$, especially in the case of a normal lattice (see Section 3).

In general, if ν is an arbitrary outer measure on the power set of X, it is very difficult to give a description of the ν-measurable sets, or even to give nontrivial classes of sets which are ν-measurable. In the case of μ, a measure, and $\nu = \mu^*$ the induced outer measure, then, of course, classes of sets which are ν-measurable are well-known. This is also the case in metric spaces with ν a metric outer measure. Here, we consider $\mu \in M(\mathcal{L})$ or $M_\sigma(\mathcal{L})$ and two associated "outer" measures $\nu = \mu'$ and $\nu = \mu''$ and attempt to obtain the ν-measurable sets. A full description can be given in case $\mu \in I(\mathcal{L})$ or $I_\sigma(\mathcal{L})$ (see Section 4), and we attempt to extend some of these results to the more general situation. We also seek necessary and sufficient conditions for various of the μ, μ', μ'' to be equal on $\mathcal{L}$ or $\mathcal{L}'$, the complementary lattice, under varying conditions on the measure and on $\mathcal{L}$.

Finally, in Section 5, we give lattice separating conditions between pairs of lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ in terms of μ', μ'' or some other "outer" measure.

We begin by giving a brief review of the basic lattice and measure theoretic terminology and notation which will be used throughout the paper. This terminology will be consistent with standard usage (see e.g. [1], [6], [7], [8], [9], [11]).

2. BACKGROUND AND NOTATION.

We shall let $\mathcal{L}$ denote a lattice of subsets of a set X and shall assume that the empty set and X are in $\mathcal{L}$. $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$. If $\mathcal{L}$ is closed under countable intersections then $\mathcal{L}$ is said to be a **δ-lattice**. $\mathcal{L}$ is said to be **normal** if whenever $A, B \in \mathcal{L}$ such that $A \cap B = \emptyset$, there exist $C, D \in \mathcal{L}$ such that $A \subset C', B \subset D'$ and $C' \cap D' = \emptyset$. $\mathcal{L}$ is **regular** if for each $x \in X$ and $A \in \mathcal{L}$ such that $x \notin A$, there exist $B, C \in \mathcal{L}$ with $x \in B', A \subset C'$ and $B' \cap C' = \emptyset$. $\mathcal{L}$ is **complement generated** if for all $L \in \mathcal{L}$, $L = \bigcap_{i=1}^{+\infty} A_i$ $\mathcal{L}$ is complement generated if for all $L \in \mathcal{L}, L = \bigcap_{i=1}^{+\infty} A'_i$, $A_i \in \mathcal{L}$. $\mathcal{L}$ is countably paracompact if whenever $\{A_i\}$ is a decreasing sequence of lattice sets in which $\bigcap_{i=1}^{+\infty} A_i = \emptyset$, there exists a decreasing sequence of $\mathcal{L}'$ sets $\{B_i'\}$ such that $A_i \subset B_i'$ for all i and $\bigcap_{i=1}^{+\infty} B_i' = \emptyset$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattices of subsets of X and $\mathcal{L}_1 \subset \mathcal{L}_2$, then **$\mathcal{L}_1$ separates $\mathcal{L}_2$** if whenever $A, B \in \mathcal{L}_2$ such that $A \cap B = \emptyset$, there exist C, $D \in \mathcal{L}_1$ such that $A \subset C$, $B \subset D$, and $C \cap D = \emptyset$; **$\mathcal{L}_1$ semiseparates $\mathcal{L}_2$** if whenever $A \in \mathcal{L}_1$ and $B \in \mathcal{L}_2$ such that $A \cap B = \emptyset$, there exists $C \in \mathcal{L}_1$ such that $B \subset C$ and $A \cap C = \emptyset$. If $\mathcal{L}_1$ separates $\mathcal{L}_2$ then $\mathcal{L}_1$ is normal if and only if $\mathcal{L}_2$ is normal.

$M(\mathcal{L})$ denotes the set of all bounded and finitely additive measures defined on $\mathcal{A}(\mathcal{L})$. Without loss of generality, we assume that these measures are non-negative. A measure μ is σ-smooth on $\mathcal{L}$ if $L_i \in \mathcal{L}$ and $L_i \downarrow \emptyset$ implies $\mu(L_i) \to 0$. $M^{\sigma}(\mathcal{L})$ will denote the set of all bounded and finitely additive measures which are σ-smooth, and hence countably additive, on $\mathcal{A}(\mathcal{L})$. If for all $A \in \mathcal{A}(\mathcal{L})$, $\mu(A) = \sup \mu(L)$, where $L \subset A$, $L \in \mathcal{L}$, then μ is said to be **$\mathcal{L}$-regular**. $M_R(\mathcal{L})$ denotes the subset of $M(\mathcal{L})$ consisting of all $\mathcal{L}$-regular measures, and $M_R^{\sigma}(\mathcal{L})$ that subset of $M_R(\mathcal{L})$ consisting of σ-smooth, $\mathcal{L}$-regular measures, i.e., $M_R^{\sigma}(\mathcal{L}) = M_R(\mathcal{L}) \cap M^{\sigma}(\mathcal{L})$. $M_{\sigma}(\mathcal{L})$ denotes those measures in $M(\mathcal{L})$ which are σ-smooth on $\mathcal{L}$. $I(\mathcal{L})$ denotes the subset of $M(\mathcal{L})$ containing precisely the $0-1$ nontrivial measures; similarly, $I_R(\mathcal{L})$, $I^{\sigma}(\mathcal{L})$, $I_R^{\sigma}(\mathcal{L})$ and $I_{\sigma}(\mathcal{L})$ denote those subsets of $M_R(\mathcal{L})$, $M^{\sigma}(\mathcal{L})$, $M_R^{\sigma}(\mathcal{L})$ and $M_{\sigma}(\mathcal{L})$ respectively, which are in $I(\mathcal{L})$. We note that there is a one-to-one correspondence between prime filters on $\mathcal{L}$ and measures in $I(\mathcal{L})$, and between $\mathcal{L}$-ultrafilters and measures in $I_R(\mathcal{L})$. Furthermore, a prime filter on $\mathcal{L}$ has the **countable intersection property** (i.e., the intersection of any countable number of prime filter set is nonempty) if and only if the corresponding measure is in $I_{\sigma}(\mathcal{L})$. If $\mu \in M(\mathcal{L})$, $S(\mu)$ denotes the **support** of μ, i.e., $S(\mu) = \cap L$ such that $L \in \mathcal{L}$ and $\mu(L) = \mu(X)$. If $\mu, \nu \in M(\mathcal{L})$ we will write $\mu \leq \nu(\mathcal{L})$, or $\mu \leq \nu$ on $\mathcal{L}$, whenever $\mu(L) \leq \nu(L)$ for all $L \in \mathcal{L}$. One can show (c.f. [10]) that if $\mu \in M(\mathcal{L})$ then there exists a $\nu \in M_R(\mathcal{L})$ such that $\mu \leq \nu(\mathcal{L})$ and $\mu(X) = \nu(X)$; if $\mathcal{L}$ is normal and $\mu \in I(\mathcal{L})$, then $\nu \in I_R(\mathcal{L})$ and ν is unique.

$$I_s(\mathcal{L}) = \{\mu \in I(\mathcal{L}) \mid \text{ if } L = \bigcap_{i=1}^{\infty} L_i,\ L,\ L_i \in \mathcal{L}, \text{ then } \mu(L) = \inf\ \mu(L_i)\}.$$

Similarly,

$$M_s(\mathcal{L}) = \{\mu \in M(\mathcal{L}) \mid \text{ if } L = \bigcap_{i=1}^{\infty} L_i,\ L,\ L_i \in \mathcal{L}, \text{ then } \mu(L) = \inf\ \mu(L_i)\}.$$

If $\mathcal{L}$ is normal and complement generated then $\mu \in I_s(\mathcal{L})$ implies $\mu \in I_R^{\sigma}(\mathcal{L})$ (c.f. [5]). Essentially the same proof shows that $\mu \in M_s(\mathcal{L})$ implies $\mu \in M_R^{\sigma}(\mathcal{L})$. If $\mathcal{L}$ is normal, $\mu \in I_R(\mathcal{L})$ and $\rho \in I_R(\mathcal{L}')$, $\rho \leq \mu(\mathcal{L})$, then $\mu(L') = \sup\ \rho(A)$, $A \subset L'$, A, $L \in \mathcal{L}$. If $\mathcal{L}_1 \subset \mathcal{L}_2$ where $\mathcal{L}_1$ separates $\mathcal{L}_2$ and if $\mu \in I_R(\mathcal{L}_1)$, $\nu \in I_R(\mathcal{L}_2)$ where ν extends μ, then ν is $\mathcal{L}_1$-regular on $\mathcal{L}'_2$, and ν is unique; furthermore, if $\mu \in I_R^{\sigma}(\mathcal{L}_1)$ then $\nu \in I_R(\mathcal{L}_2) \cap I_{\sigma}(\mathcal{L}'_2)$.

3. EXTENSIONS OF SOME RESULTS TO MORE GENERAL MEASURES.

It is interesting to note how results of Section 2 generalize and extend to measures which are not zero-one, i.e., to elements of $M(\mathcal{L})$, $M_R(\mathcal{L})$ etc. We elaborate on a number of these below.

LEMMA 3.1. Let $\mathcal{L}$ be normal, $\mu \in M(\mathcal{L})$, $\mu \leq \nu(\mathcal{L})$ where $\nu \in M_R(\mathcal{L})$ and $\mu(X) = \nu(X)$. Then for $L \in \mathcal{L}, \nu(L') = \sup\ \mu(A), A \subset L', A \in \mathcal{L}$.

PROOF. Let $\nu \in M_R(\mathcal{L})$. Then for $\varepsilon > 0$ and $L \in \mathcal{L}$, there exists $L_1 \in \mathcal{L}, L_1 \subset L'$ such that $\nu(L') - \nu(L_1) < \varepsilon$. Since $\mathcal{L}$ is normal, there exist A, $B \in \mathcal{L}$ such that $L_1 \subset A' \subset B \subset L'$. Therefore $\mu(B) \geq \mu(A') \geq \nu(A') \geq \nu(L_1)$. Therefore, $\nu(L') - \mu(B) < \varepsilon$.

THEOREM 3.1. Suppose $\mathcal{L}$ is normal, $\mu \in M(\mathcal{L})$, ν_1, $\nu_2 \in M_R(\mathcal{L})$, $\mu \leq \nu_1(\mathcal{L})$, $\mu \leq \nu_2(\mathcal{L})$ and $\mu(X) = \nu_1(X) = \nu_2(X)$. Then $\nu_1 = \nu_2$.

PROOF. It follows from Lemma 3.1 that $\nu_1(L') = \nu_2(L')$ for all $L \in \mathcal{L}$. Therefore, $\nu_1 = \nu_2$.

Let $\mu \in M(\mathcal{L})$. Define $\rho(E) = sup\ \mu(L')$, $L' \subset E$, $E \subset X$ and $L \in \mathcal{L}$. Our next result shows that the supports of μ and ρ are equal if $\mathcal{L}$ is regular. The definition of support is extended to ρ in an obvious way.

THEOREM 3.2. If $\mathcal{L}$ is regular then $S(\mu) = S(\rho)$.

PROOF. Since $\rho(L) \leq \mu(L)$ for all $L \in \mathcal{L}$, $S(\mu) \subset S(\rho)$.

Suppose there exists an $x \in S(\rho)$ and $x \notin S(\mu)$. If $\rho(L) = \rho(X)$, and $L \in \mathcal{L}$ then $x \in L$. But there exists $A \in \mathcal{L}$ such that $\mu(A) = \mu(X)$ and $x \notin A$. Since $\mathcal{L}$ is regular, there exist L_1, $L_2 \in \mathcal{L}$ such that $x \in L'_1 \subset L_2 \subset A'$. Therefore $\mu(L'_2) \geq \mu(A) = \mu(X)$. It follows that $\rho(L'_2) = \rho(X)$. Therefore $\rho(L_1) = \rho(X)$ and hence $x \in L_1$, a contradiction.

DEFINITION. $\mathcal{L}$ is **almost countably compact** (a.c.c.) if $\mu \in I_R(\mathcal{L}')$ implies $\mu \in I_\sigma(\mathcal{L})$.

We now show that if $\mathcal{L}$ is a.c.c. then its defining condition holds for general measures.

THEOREM 3.3. Suppose $\mathcal{L}$ is a.c.c. Then $\mu \in M_R(\mathcal{L}')$ implies $\mu \in M_\sigma(\mathcal{L})$.

PROOF. Let $A_i \in \mathcal{L}$, $A'_i \downarrow \emptyset$, $A'_i \neq \emptyset$, let $\mu \in M_R(\mathcal{L}')$, and let $\mathcal{H} = \{B \in \mathcal{L} \mid A'_i \subset B \text{ for some } i\}$. Now $\{A'_i\}$ can be enlarged to an $\mathcal{L}'$ ultrafilter. Therefore, there exists $\nu \in I_R(\mathcal{L}')$ such that $\nu(A'_i) = 1$ for all i. Since $\mathcal{L}$ is a.c.c., $\nu \in I_\sigma(\mathcal{L})$. Thus $\mathcal{H}$ has the countable intersection property. Suppose $\mathcal{L}_i \downarrow \emptyset$, $L_i \in \mathcal{L}$, and suppose $\mu(L_i) > \varepsilon > 0$ for all i. Since $\mu \in M_R(\mathcal{L}')$, there exist $A'_i \subset L_i, A_i \in \mathcal{L}$, and $A'_i \downarrow \emptyset$, such that $\mu(A'_i) > \varepsilon/2$ for all i. Now $A'_i \neq \emptyset$ for any i and $L_i \in \mathcal{H}$. Therefore $\mathcal{H}$ does not have the countable intersection property, a contradiction.

THEOREM 3.4. Suppose $\mathcal{L}_1 \subset \mathcal{L}_2$ where $\mathcal{L}_1$ separates $\mathcal{L}_2$. Let $\mu \in M_R(\mathcal{L}_1)$, $\nu \in M_R(\mathcal{L}_2)$ and let ν extend μ. Then the following are true:

a) ν is $\mathcal{L}_1$-regular on $\mathcal{L}'_2$.

b) If $\nu_1 \in M_R(\mathcal{L}_2)$ and ν_1 extends μ then $\nu = \nu_1$.

c) $\mu \in M_R{}^\sigma(\mathcal{L}_1)$ implies $\nu \in M_R(\mathcal{L}_2) \cap M_\sigma(\mathcal{L}'_2)$.

d) $\mu \in M_R{}^\sigma(\mathcal{L}_1)$ and $M_\sigma(\mathcal{L}'_2) \subset M_\sigma(\mathcal{L}_2)$ implies $\nu \in M_R{}^\sigma(\mathcal{L}_2)$.

PROOF. a) Let $L_2 \in \mathcal{L}_2$. Since $v \in M_R(\mathcal{L}_2)$, for any $\varepsilon > 0$, there exists $A_2 \in \mathcal{L}_2$ such that $\nu(L'_2) < \nu(A_2) + \varepsilon$, $A_2 \subset L'_2$. Since $\mathcal{L}_1$ separates $\mathcal{L}_2$, there exist L_1, $A_1 \in \mathcal{L}_1$ such that $L_2 \subset L_1$, $A_2 \subset A_1$, and $L_1 \cap A_1 = \emptyset$, i.e., $A_2 \subset A_1 \subset L'_1 \subset L'_2$. Therefore, $\nu(L'_2) < \nu(A_2) + \varepsilon < \nu(A_1) + \varepsilon$.

b) If ν and $\nu_1 \in M_R(\mathcal{L}_2)$ and are extensions of μ, it follows from part a) that $\nu(L'_2) = \nu_1(L'_2)$ for all $L_2 \in \mathcal{L}_2$. Therefore $\nu = \nu_1$.

c) Let $\nu \in M_R(\mathcal{L}_2)$ be an extension of $\mu \in M_R{}^\sigma(\mathcal{L}_1)$, let $A'_i \downarrow \emptyset$, $A_i \in \mathcal{L}_2$ and let $\varepsilon > 0$ be given. Since ν is $\mathcal{L}_1$-regular on $\mathcal{L}'_2$, there exist $B_i \subset A'_i$, $B_i \downarrow \emptyset$, $B_i \in \mathcal{L}_1$, such that $\nu(A'_i) < \mu(B_i) + \varepsilon/2$. Since $\mu \in M_R{}^\sigma(\mathcal{L}_1)$, there exists $B_N \in \mathcal{L}_1$ such that $\mu(B_N) < \varepsilon/2$. Thus $\nu(A'_N) < \varepsilon$ and hence $\nu \in M_\sigma(\mathcal{L}'_2)$.

d) From c), $\nu \in M_R(\mathcal{L}_2) \cap M_\sigma(\mathcal{L}'_2) \subset M_R(\mathcal{L}_2) \cap M_\sigma(\mathcal{L}_2) = M_R{}^\sigma(\mathcal{L}_2)$.

THEOREM 3.5. Suppose $\mathcal{L}$ is normal. Let $\mu \in M_\sigma(\mathcal{L})$, $\nu \in M_R(\mathcal{L})$, $\mu \leq \nu(\mathcal{L})$ and $\mu(X) = \nu(X)$. Then $\nu \in M_\sigma(\mathcal{L}')$.

PROOF. Let $L'_i \downarrow \emptyset$, $L_i \in \mathcal{L}$ for all i and let $\varepsilon > 0$.

Since $\nu \in M_R(\mathcal{L})$, there exist $A_i \in \mathcal{L}$ such that $A_i \downarrow \emptyset$, $A_i \subset L'_i$ and $\nu(L'_i) < \nu(A_i) + \varepsilon/2$. Since $\mathcal{L}$ is normal, there exist B_i, $C_i \in \mathcal{L}$ such that $A_i \subset B'_i \subset C_i \subset L'_i$ for all i, where $B'_i \downarrow \emptyset$ and $C_i \downarrow \emptyset$. Therefore, $\nu(L'_i) < \nu(B'_i) + \varepsilon/2 \leq \mu(B'_i) + \varepsilon/2 \leq \leq \mu(C_i) + \varepsilon/2$.

Since $\mu \in M_\sigma(\mathcal{L})$, there exists C_N such that $\mu(C_N) < \varepsilon/2$. It follows that $\nu(L'_N) < \varepsilon$.

4. ASSOCIATED OUTER MEASURES AND LATTICE PROPERTIES.

In this section we introduce the associated "outer" measures μ' and μ'' and compare the behavior of $\mu \in I(\mathcal{L})$ or $I_\sigma(\mathcal{L})$ with that of $\mu \in M(\mathcal{L})$ or $M_\sigma(\mathcal{L})$. We consider relationships between μ' and μ'' under added lattice assumptions.

Let $\mu \in M(\mathcal{L})$ and let $E \subset X$. Define $\mu'(E) = inf\ \mu(L')$, where inf is taken over all L' sets such that $E \subset L'$, $L \in \mathcal{L}$.

Let $\mu \in M_\sigma(\mathcal{L})$ and let $E \subset X$. Define $\mu''(E) = inf \sum_{i=1}^{\infty} \mu(L'_i)$, where inf is taken over all L'_i sets such that $E \subset \bigcup_{i=1}^{\infty} L'_i, L_i \in \mathcal{L}$.

It is evident that μ' is a finitely subadditive "outer" measure, i.e., μ' has the defining properties of an outer measure with the exception that it is only finitely subadditive. On the other hand, μ'' is an outer measure. For the following reason, we assume $\mu \in M_\sigma(\mathcal{L})$ when defining μ''. If μ'' were defined for all $\mu \in M(\mathcal{L})$ then in particular, if $\mu \in I(\mathcal{L})$ and $\mu \notin I_\sigma(\mathcal{L})$ then $\mu'' \equiv 0$.

THEOREM 4.1. Let $\mu \in M_\sigma(\mathcal{L})$. Then

a) $\mu'' \leq \mu'$ everywhere

b) $\mu''(X) = \mu(X)$

c) $\mu \leq \mu''$ on $\mathcal{L}$.

PROOF. a) Clear.

b) Clearly $\mu''(X) \leq \mu(X)$. Let $\varepsilon > 0$ be given. Then there exist $L_i \in \mathcal{L}$ such that $X = \bigcup_{i=1}^{\infty} L'_i$, and $\mu''(X) + \varepsilon > \sum_{i=1}^{\infty} \mu(L'_i)$. Let $A'_N = \bigcup_{i=1}^{N} L'_i$. Then, $\sum_{i=1}^{\infty} \mu(L'_i) \geq \sum_{i=1}^{N} \mu(L'_i) \geq \mu(A'_N) = \mu(X) - \mu(A_N)$. Since $A_N \downarrow \emptyset$, it follows that $\mu''(X) + \varepsilon > \mu(X)$.

c) Let $L \in \mathcal{L}$. Then $\mu''(L) \geq \mu''(X) - \mu''(L') = \mu(X) - \mu''(L') \geq \mu(X) - \mu(L') = \mu(L)$.

REMARK. Let $E \subset X$ and let $\mu^{\cdot}(E) = inf\ \mu(A)$, $E \subset A$, $A \in \mathcal{A}(\mathcal{L})$.

We note that under the assumptions of Theorem 3.4, $\nu = \mu^{\cdot} = \mu'$ on $\mathcal{L}_2$.

The following theorem concerning supports is a generalization of a result in [5]. We omit the proof since it is essentially the same proof given there (c.f. Theorem 4.10 [5]) and note a corollary pertaining to μ'.

THEOREM 4.2. Let $\mu \in M(\mathcal{L})$ where $\mathcal{L}$ is regular. Let τ be a monotone set function defined on any collection of sets containing $\mathcal{A}(\mathcal{L})$, where $\tau \geq 0$, $\mu \leq \tau$ on $\mathcal{L}$, $\tau \leq \mu$ on $\mathcal{L}'$, and $\tau(X) = \mu(X)$. Then $S(\tau) = S(\mu)$.

COROLLARY 4.1. Let $\mu \in M(\mathcal{L})$, and let $\mathcal{L}$ be regular. Then $S(\mu') = S(\mu)$.

Let $\mathcal{T}_{\mu'}$ denote the collection of μ'-measurable sets, and let $\mathcal{T}_{\mu''}$ denote the collection of μ''-measurable sets. Theorem 4.3 presents a classification of these sets for 0-1 measures.

THEOREM 4.3. a) Let $\mu \in I(\mathcal{L})$.

Then $\mathcal{T}_{\mu'} = \{E \subset X \mid E \supset L \text{ or } E' \supset L, L \in \mathcal{L}, \mu(L) = 1\}$.

b) Let $\mu \in I_\sigma(\mathcal{L})$. Then $\mathcal{T}_{\mu''} = \{E \subset X \mid E \supset \bigcap_{i=1}^{\infty} L_i \text{ or } E' \supset \bigcap_{i=1}^{\infty} L_i,\ \mu(L_i) = 1,\ L_i \in \mathcal{L} \text{ for all } i\}$.

PROOF. a) Let $E \subset X$, and $L \in \mathcal{L}$.

If $L \subset E$ and $\mu(L) = 1$ then $\mu'(E) = 1$ and $\mu'(E') = 0$.

If $L \subset E'$ and $\mu(L) = 1$ then $\mu'(E') = 1$ and $\mu'(E) = 0$.

In either case, $E \in \mathcal{T}_{\mu'}$.

Conversely, suppose $E \in \mathcal{T}_{\mu'}$. Then $\mu'(E) + \mu'(E') = 1$.

If $\mu'(E) = 0$ then there exists $L \in \mathcal{L}$ such that $E \subset L'$ and $\mu(L') = 0$. Thus $\mu(L) = 1$.

If $\mu'(E') = 0$ then there exists $L \in \mathcal{L}$ such that $L \subset E$ and $\mu(L) = 1$.

b) Let $E \subset X, L_i \in \mathcal{L}, E \supset \bigcap_{i=1}^{\infty} L_i$ and $\mu(L_i) = 1$ for all i.

Then $E' \subset \bigcup_{i=1}^{\infty} L'_i$ and $\mu(L'_i) = 0$ for all i. Therefore $\mu''(E') = 0$ and $E \in \mathfrak{T}_{\mu''}$. A symmetric argument proves that $E \in \mathfrak{T}_{\mu''}$ if $E' \supset \bigcap_{i=1}^{\infty} L_i$.
Conversely, let $E \subset X$ and suppose $\mu''(A) = \mu''(A \cap E) + \mu''(A \cap E')$ for all $A \subset X$. In particular, $1 = \mu''(X) = \mu''(E) + \mu''(E')$. If $\mu''(E) = 0$ then there exist $L_i \in \mathcal{L}$ such that $E \subset \bigcup_{i=1}^{\infty} L'_i$ and $\mu(L_i) = 1$ for all i. Similarly, $\mu''(E') = 0$ implies $E \supset \bigcap_{i=1}^{\infty} L_i$ and $\mu(L_i) = 1$ for all i.

The sets $I_s(\mathcal{L})$ and $M_s(\mathcal{L})$ provide a framework from which many of the remaining theorems of this section rely, particularly with respect to results concerning μ'' and the μ''-measurable sets.

REMARK. If $\mu \in M(\mathcal{L})$ then $E \in \mathfrak{T}_{\mu'}$ iff $\mu'(A') \geq \mu'(A' \cap E) + \mu'(A' \cap E')$ for all $A' \in \mathcal{L}'$.

LEMMA 4.1. Let $\mu \in M(\mathcal{L})$ and let $E \subset X$. Then $E \in \mathfrak{T}_{\mu'}$ if and only if $\mu'(E) = \sup \mu(L)$ where $L \subset E$ and $L \in \mathcal{L}$.

PROOF. Suppose $\mu'(E) = \sup \mu(L)$. Let $\varepsilon > 0$ be given. Then there exists $L \in \mathcal{L}$ such that $L \subset E$ and $\mu'(E) - \mu(L) < \varepsilon/2$. Similarly, by definition of μ', there exists $B' \supset E$ such that $B \in \mathcal{L}$ and $\mu(B') - \mu'(E) < \varepsilon/2$. Therefore, $\mu(B') - \mu(L) < \varepsilon$ and $L \subset E \subset B'$. Let $A' \in \mathcal{L}'$. Now $\mu'(A' \cap E) \leq \mu(A' \cap B') = \mu(A') + \mu(B') - \mu(A' \cup B') < \mu(A') + \mu(L) + \varepsilon - \mu(A' \cup L) = \mu(A' \cap L) + \varepsilon$. Therefore, $\mu(A' \cap L) > \mu'(A' \cap E) - \varepsilon$. It follows that, $\mu(A') = \mu(A' \cap L) + \mu(A' \cap L') \geq \mu(A' \cap L) + \mu'(A' \cap E') > \mu'(A' \cap E) - \varepsilon + \mu'(A' \cap E')$ and hence $E \in \mathfrak{T}_{\mu'}$. Conversely, suppose $E \in \mathfrak{T}_{\mu'}$. Then $\mu'(E') = \mu(X) - \mu'(E)$. Also, if $\varepsilon > 0$, there exists $L \in \mathcal{L}$ such that $L' \supset E'$ and $\mu(L') - \mu'(E') < \varepsilon$. Therefore, $\mu'(E) - \mu(L) < \varepsilon$.

THEOREM 4.4. a) Let $\mu \in M(\mathcal{L})$. Then $\mathfrak{T}_{\mu'} \cap \mathcal{L} = \{L \in \mathcal{L} \mid \mu'(L) = \mu(L)\}$.
b) Let $\mu \in I_\sigma(\mathcal{L})$. Then $\mathfrak{T}_{\mu''} \cap \mathcal{L} = \{L \in \mathcal{L} \mid \mu''(L) = \mu(L)\}$ iff $\mu \in I_s(\mathcal{L})$.

PROOF. a) Clearly follows form Lemma 4.1.
b) Let $\mathcal{L}_{\mu''} = \{L \in \mathcal{L} \mid \mu''(L) = \mu(L)\}$. Suppose $\mathfrak{T}_{\mu''} \cap \mathcal{L} = \mathcal{L}_{\mu''}$. Let $L_i \downarrow L$, where $L_i L \in \mathcal{L}$. Suppose $\mu(L_i) = 1$ for all i. Then $L \in \mathfrak{T}_{\mu''} \cap \mathcal{L}$. If $\mu(L) = 0$ then $\mu''(L) = 0$. Therefore there exist $A_i \in \mathcal{L}, A_i \downarrow, L \subset \bigcup_{i=1}^{\infty} A'_i$ and $\mu(A'_i) = 0$ for all i. Therefore, $(L_i \cap A_i) \downarrow \emptyset$. But $\mu(L_i \cap A_i) = 1$ for all i, a contradiction since $\mu \in I_\sigma(\mathcal{L})$. Conversely, suppose $\mu \in I_s(\mathcal{L})$ and $L \in \mathfrak{T}_{\mu''} \cap \mathcal{L}$. Let $L, L_i \in \mathcal{L}$.

CASE 1: $\bigcap_{i=1}^{\infty} L_i \subset L$ and $\mu(L_i) = 1$ for all i.
Then $\bigcap_{i=1}^{\infty} (L \cap L_i) = L$ and we may assume $(L \cap L_i) \downarrow$.
Since $\mu \in I_s(\mathcal{L}), \mu(L) = 1$. Therefore $\mu''(L) = 1$.

CASE 2: $\bigcap_{i=1}^{\infty} L_i \subset L'$ and $\mu(L_i) = 1$ for all i.
Then $\mu(L'_i) = 0$ for all i and $L \subset \bigcup_{i=1}^{\infty} L'_i$. Therefore $\mu''(L) = 0$ and hence $\mu(L) = 0$.
Thus $\mathfrak{T}_{\mu''} \cap \mathcal{L} \subset \mathcal{L}_{\mu''}$. But clearly, $\mathfrak{T}_{\mu''} \cap \mathcal{L} \supset \mathcal{L}_{\mu''}$.

THEOREM 4.5. Suppose $\mu \in M(\mathcal{L})$. Then $\mathcal{L} \subset \mathfrak{T}_{\mu'}$ iff $\mu \in M_R(\mathcal{L})$.

PROOF. Clear.

We next investigate conditions which guarantee that μ' and μ'', or $\mathfrak{T}_{\mu'}$ and $\mathfrak{T}_{\mu''}$ are equal. Our first result concerns the equality of these outer measures on $\mathcal{L}'$.

THEOREM 4.6. Let $\mu \in M_\sigma(\mathcal{L})$. If $\mu \in M_s(\mathcal{L})$ then $\mu' = \mu''$ on $\mathcal{L}'$.

PROOF. We know that $\mu'' \leq \mu'$ on $\mathcal{L}'$. Let $\varepsilon > 0$ be given and let $L \in \mathcal{L}$. Then there exist $L_i \in \mathcal{L}$ such that $L' \subset \bigcup_{i=1}^{\infty} L'_i$ and $\mu''(L') + \varepsilon/2 > \sum_{i=1}^{\infty} \mu(L'_i)$. (We assume $L'_i \uparrow$.) Therefore, $L = \bigcap_{i=1}^{\infty} (L_i \cup L)$ and $(L_i \cup L) \downarrow L$. Since $\mu \in M_s(\mathcal{L})$, there exists L_N such that $\mu(L_N \cup L) - \mu(L) < \varepsilon/2$ for some N, or equivalently, $\mu(L'_N \cap L') > \mu(L') - \varepsilon/2$.
Therefore,

$$\mu''(L') + \varepsilon/2 > \sum_{i=1}^{\infty} \mu(L'_i) \geq \mu(L'_N) \geq \mu(L'_N \cap L') > \mu(L') - \varepsilon/2$$

from which it follows that $\mu''(L') = \mu(L') = \mu'(L')$.

THEOREM 4.7. Let $\mu \in M_\sigma(\mathcal{L})$ and let $\mathcal{L}$ be normal. Suppose either of the following two conditions is true:

i) $\mathcal{L}$ is countably paracompact.

ii) $\mathcal{L}$ is a δ-lattice.

Then $\mu' = \mu''$ on $\mathcal{L}$.

PROOF. i) Since $\mathcal{L}$ is normal and countably paracompact, there exists $\nu \in M_R{}^\sigma(\mathcal{L})$ such that $\mu \le \nu$ on $\mathcal{L}$ and $\mu(X) = \nu(X)$. (c.f. [11]) Therefore, on $\mathcal{L}$, $\nu = \nu' = \nu'' \le \mu'' \le \mu'$. Also, since $\mathcal{L}$ is normal, $\nu' = \mu'$ on $\mathcal{L}$.

ii) Let $\varepsilon > 0$ be given and let $L \in \mathcal{L}$. There exist $L_i \downarrow$, $L \subset \bigcup_{i=1}^{\infty} L'_i$ and $\mu''(L) + \varepsilon > \sum_{i=1}^{\infty} \mu(L'_i)$. Since $\mathcal{L}$ is δ, if $A' = \bigcup_{i=1}^{\infty} L'_i$ then $A \in \mathcal{L}$. Since $\mathcal{L}$ is normal, there exist $B, C \in \mathcal{L}$ such that $L \subset B' \subset C \subset A'$. Therefore,

$$\mu'(L) \le \mu(B') \le \mu(C) \le \mu''(C) \le \mu''(A') \le \sum_{i=1}^{\infty} \mu(L'_i) < \mu''(L) + \varepsilon.$$

THEOREM 4.8. Suppose $\mathcal{L}$ is δ and $\mu \in M_s(\mathcal{L})$. Then

a) $\mu(\bigcup_{i=1}^{\infty} L'_i) \le \sum_{i=1}^{\infty} \mu(L'_i), L_i \in \mathcal{L}$.

b) $\mu' = \mu''$ everywhere

c) $\mathcal{T}_{\mu'} = \mathcal{T}_{\mu''}$.

The following corollary follows immediately from Theorem 4.5 and Theorem 4.8 c).

COROLLARY 4.2. Suppose $\mathcal{L}$ is δ, $\mu \in M_s(\mathcal{L})$ and $\mathcal{L} \subset \mathcal{T}_{\mu''}$. Then $\mu \in M_R(\mathcal{L})$.

The following theorem shows that set inclusion of μ'-measurable sets is preserved under inequalities with respect to the lattice $\mathcal{L}$.

THEOREM 4.9. Let $\mu, \nu \in M(\mathcal{L}), \mu \le \nu$ on $\mathcal{L}$ and $\mu(X) = \nu(X)$. Then $\mathcal{T}_{\mu'} \subset \mathcal{T}_{\nu'}$.

PROOF. $E \in \mathcal{T}_{\mu'}$ implies $\mu(E) = \sup\, \mu(L) \le \sup\, \nu(L) \le \nu'(E)$, $L \subset E$, $L \in \mathcal{L}$. But since $\mu \le \nu$ on $\mathcal{L}$, $\nu \le \mu$ on $\mathcal{L}'$ and hence $\nu'(E) \le \mu'(E)$.

We next note some extensions of some results which are known for zero-one measures that require the notion of a regular outer measure. We begin by defining this concept and list some consequences.

Let ν be a finitely subadditive outer measure. Then ν is **regular** if for every $G \subset X$, there exists $E \in \mathcal{T}_\nu$ such that $G \subset E$ and $\nu(G) = \nu(E)$.

The following properties are noted for completeness:

i) Let ν be a regular outer measure. If $E_i\uparrow$, $E_i \subset X$, then $\nu(\lim E_i) = \lim\, \nu(E_i)$.

ii) Let ν be a regular, finitely subadditive, outer measure. Then $E \in \mathcal{T}_\nu$ iff $\nu(X) = \nu(E) + \nu(E')$.

REMARK. Clearly, i) is not true if ν is a finitely subadditive outer measure. For example, let $\mu \in I(\mathcal{L}) - I_\sigma(\mathcal{L})$. Then there exist $L_i \in \mathcal{L}$ such that $L_i \downarrow \emptyset$ and $\mu(L_i) = 1$ for all i. Therefore, $\mu'(X) = \mu(X) = 1$ but $\mu'(L_i) = 0$ for all i.

We now show that the converse of Theorem 4.6 is valid when μ'' is regular.

THEOREM 4.10. Let $\mu \in M_\sigma(\mathcal{L})$. If $\mu' = \mu''$ on $\mathcal{L}'$ and if μ'' is regular then $\mu \in M_s(\mathcal{L})$.

PROOF. Suppose $\mu' = \mu''$ on $\mathcal{L}'$. Let $L_i \downarrow L$, L_i, $L \in \mathcal{L}$. Assume there exists $\varepsilon > 0$ such that $\mu(L) + \varepsilon < \mu(L_i)$ for all i. Then $\mu(L') - \varepsilon > \mu''(L'_i)$ for all i and hence $\mu(L') - \varepsilon \ge \lim\ \mu''(L'_i)$ $= \mu''(\lim L'_i) = \mu''(L') = \mu(L')$, a contradiction.

We end this section with some further consequences of regular outer measures which are stated without proof in the following theorem.

THEOREM 4.11. Let $\mu \in M_\sigma(\mathcal{L})$ and let μ'' be regular. Then the following hold:

a) $\mathcal{T}_{\mu'} \subset \mathcal{T}_{\mu''}$.

b) If $\mu = \mu''$ on $\mathcal{L}$ then $\mu \in M^\sigma(\mathcal{L})$.

Clearly, if $\mu \in I(\mathcal{L})$ then μ' is regular. Similarly, if $\mu \in I_\sigma(\mathcal{L})$ then μ'' is regular. Therefore, the consequences of Theorems 4.10 and 4.11 are valid if $\mu \in I_\sigma(\mathcal{L})$. It is interesting to note that no explicit use of the regular assumption of outer measures is required in proving this case.

5. ASSOCIATED OUTER MEASURES AND LATTICE SEPARATION.

In this section we investigate the effect of further lattice assumptions on various measures and for this purpose introduce the class of measures $M_W(\mathcal{L})$. Also, we investigate the effect of assumptions concerning "outer" measures on lattice separation properties.

For $E \subset X$ define $\tilde{\mu}(E) = \inf\ \mu(L)$ where $E \subset L, L \in \mathcal{L}$. We note that $\tilde{\mu}$ is a finitely subadditive outer measure.

LEMMA 5.1. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ and suppose $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$. If $\mu \in M(\mathcal{L}_1)$ then $\mu' \geq \tilde{\mu}$ on $\mathcal{L}_2$.

PROOF. Let $\mu \in M(\mathcal{L}_1)$, $L_2 \in \mathcal{L}_2$ and $\varepsilon > 0$. Then there exists $L'_1 \supset L_2$, $L_1 \in \mathcal{L}_1$ such that $\mu'(L_2) > \mu(L'_1) - \varepsilon$. If $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$, there exists $A_1 \in \mathcal{L}_1$ such that $L_2 \subset A_1 \subset L'_1$.
Thus,

$$\mu'(L_2) > \mu(L'_1) - \varepsilon \geq \mu(A_1) - \varepsilon \geq \tilde{\mu}(L_2) - \varepsilon.$$

Therefore $\mu' \geq \tilde{\mu}$ on $\mathcal{L}_2$.

LEMMA 5.2. Let $\mathcal{L}_1 \subset \mathcal{L}_2$. If $\mu \in M_R(\mathcal{L}_1)$ then $\tilde{\mu} \geq \mu'$ on $\mathcal{L}_2$.

PROOF. Let $\varepsilon > 0$ be given and let $L_2 \in \mathcal{L}_2$. Then there exists $L_1 \in \mathcal{L}_1$ such that $L_1 \supset L_2$ and $\tilde{\mu}(L_2) > \mu(L_1) - \varepsilon$. Since $\mu \in M_R(\mathcal{L}_1), \mu = \mu'$ on $\mathcal{L}_1$. Thus, $\tilde{\mu}(L_2) > \mu(L_1) - \varepsilon = \mu'(L_1) - \varepsilon \geq \mu'(L_2) - \varepsilon$. Therefore, $\tilde{\mu} \leq \mu'$ on $\mathcal{L}_2$.

Combining the results from Lemmas 5.1 and 5.2 we have,

THEOREM 5.1. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ and suppose $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$. If $\mu \in M_R(\mathcal{L}_1)$ then $\mu' = \tilde{\mu}$ on $\mathcal{L}_2$.

The following theorem gives conditions which preserve inequalities of measures when extended to super-lattices.

THEOREM 5.2. Suppose $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ separates $\mathcal{L}_2$. Let $\mu \leq \nu(\mathcal{L}_1)$ where $\mu \in M(\mathcal{L}_1), \nu \in M_R(\mathcal{L}_1)$, and let τ and λ be extensions of μ and ν to $M(\mathcal{L}_2)$ and $M_R(\mathcal{L}_2)$ respectively. Then $\tau \leq \lambda(\mathcal{L}_2)$.

PROOF. Suppose there exists $L_2 \in \mathcal{L}_2$ such that $\tau(L_2) > \lambda(L_2)$. Let $\varepsilon > 0$ and let $\tau(L_2) - \lambda(L_2) > \varepsilon$. Since $\lambda \in M_R(\mathcal{L}_2)$, there exists $A'_2 \supset L_2$, $A_2 \in \mathcal{L}_2$ such that $\lambda(L_2) + \varepsilon > \lambda(A'_2)$. Since $\mathcal{L}_1$ separates $\mathcal{L}_2$, there exist $L_1, A_1 \in \mathcal{L}_1$ such that $L_2 \subset L_1 \subset A'_1 \subset A'_2$. Therefore,

$$\mu(L_1) = \tau(L_1) \geq \tau(L_2) > \lambda(L_2) + \varepsilon > \lambda(A'_2) \geq \lambda(L_1) \doteq \nu(L_1),$$

a contradiction.

We now show that semiseparation is a sufficient condition to preserve equality of the outer measures μ', μ'' on super-lattices when the given lattice is δ.

THEOREM 5.3. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ where $\mathcal{L}_1$ is δ, $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$, and let $\mu \in M_\sigma(\mathcal{L}_1)$. If $\mu' = \mu''$ on $\mathcal{L}_1$ then $\mu' = \mu''$ on $\mathcal{L}_2$.

PROOF. If $\varepsilon > 0$ and $L_2 \in \mathcal{L}_2$, there exist $L_i \in \mathcal{L}_1$ such that $\bigcup_{i=1}^{\infty} L'_i \supset L_2$ and $\mu''(L_2) + \varepsilon > \sum_{i=1}^{\infty} \mu(L'_1)$. Since $\mathcal{L}_1$ is δ, $A'_1 = \bigcup_{i=1}^{\infty} L'_i, A_1 \in \mathcal{L}_1$.
If $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$ there exists B_1 in $\mathcal{L}_1$ such that $L_2 \subset B_1 \subset A'_1$. Now,

$$\mu'(L_2) \leq \mu'(B_1) = \mu''(B_1) \leq \mu''(A'_1) \leq \sum_{i=1}^{\infty} \mu(L'_i) < \mu''(L_2) + \varepsilon.$$

Therefore, $\mu' \leq \mu''$ on $\mathcal{L}_2$. Since $\mu' \geq \mu''$ everywhere, the conclusion follows.

Let $M_W(\mathcal{L}) = \{\mu \in M(\mathcal{L}) \mid \mu(L') = \sup\ \mu'(A),\ L,\ A \in \mathcal{L} \text{ and } A \subset L'\}$.

Clearly, $M_R(\mathcal{L}) \subset M_W(\mathcal{L})$. We now prove that $M_R(\mathcal{L}) = M_W(\mathcal{L})$ when $\mathcal{L}$ is normal.

THEOREM 5.4. Suppose $\mathcal{L}$ is normal. Then $\mu \in M_W(\mathcal{L})$ implies that $\mu \in M_R(\mathcal{L})$.

PROOF. Let $\varepsilon > 0$ and $L \in \mathcal{L}$. Since $\mu \in M_W(\mathcal{L})$, there exists $L_1 \in \mathcal{L}$ such that $L_1 \subset L'$ and $\mu(L') - \mu'(L_1) < \varepsilon$. Since $\mathcal{L}$ is normal, there exist A, $B \in \mathcal{L}$ such that $L_1 \subset A' \subset B \subset L'$.
Therefore, $\mu'(L_1) \leq \mu'(A') = \mu(A') \leq \mu(B) \leq \mu'(B) \leq \mu(L')$ and hence, $\mu(L') - \mu(B) < \varepsilon$. It follows that $\mu \in M_R(\mathcal{L})$.

The following example shows that the converse of Theorem 5.4 is not true.

EXAMPLE 5.1. Let A, $B \subset X$, $A \cup B \neq X$ and $A \cap B = \emptyset$. Let $\mathcal{L} = \{\emptyset, A, B, A \cup B, X\}$. Clearly $\mathcal{L}$ is not normal, but $M_W(\mathcal{L}) = M_R(\mathcal{L})$.

THEOREM 5.5. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ and let $\mathcal{L}_1$ semiseparate $\mathcal{L}_2$. If $\nu \in M_W(\mathcal{L}_2)$ and if μ is the restriction of ν to $\mathcal{A}(\mathcal{L}_1)$ then $\mu \in M_W(\mathcal{L}_1)$.

PROOF. Let $\varepsilon > 0$ and let $L_1 \in \mathcal{L}_1$. Then there exists $L_2 \subset L'_1$ such that $\mu(L'_1) - \varepsilon < \nu'(L_2)$. Since $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$, there exists $A_1 \in \mathcal{L}_1$ such that $L_2 \subset A_1 \subset L'_1$. Clearly, $\nu'(L_2) \leq \mu'(L_2) \leq \mu'(A_1)$. Therefore, $\mu(L'_1) - \varepsilon < \mu'(A_1)$.

REFERENCES

1. ALEXANDROFF, A.D., Additive set functions in abstract spaces, Mat. Sb., (N.S.) 8, 50 (1940), 307-348.
2. CAMACHO, JR., J., Extensions of lattice regular measures with applications, J. Indian Math. Soc. 54 (1989), 233-244.
3. EID, G., On normal lattices and Wallman spaces, Inter. J. of Math. and Math. Sci. Vol. 13, No. 1 (1990), 31-38.
4. FROLIK, Z., Prime filters with the C.I.P., Comm. Math. Univ. Carolinae 13, 3 (1972), 553-573.
5. GRASSI, P., Measure characterizations and properties of normal and regular lattices, Inter. J. of Math. and Math. Sci. Vol. 14, No. 3 (1991), 385-392.
6. GRASSI, P., On subspaces of replete and measure replete spaces, Canad. Math. Bull. 1, 27 (1984), 58-64.
7. GRASSI, P., An adjoint map between regular measures and its applications, Jour. Ind. Math. Soc. 48 (1984), 19-31.
8. HUERTA, C., Notions of compactness on the lattice and on the point set in terms of measure, Ann. Sc. Math., Quebec, 13, 1 (1989), 49-52.
9. NÖBELING, G., Grundlagen der Analytischen Topologie, Springer-Verlag, Berlin, 1954.
10. SZETO, M., On maximal measures with respect to a lattice, Measure Theory and its Applications, 1980, Northern Illinois University, 277-282.
11. SZETO, M., Measure repleteness and mapping preservations, Jour. Ind. Math. Soc. 43 (1979), 35-52.